Yield Curve Modeling

Yield Curve Building with Bonds

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- Bootstrap Method
- Regression Techniques
- Building Emerging Market Yield Curves
- Iterative Methods

Bootstrap Method for Bonds

Today | Year 1 | Year 2 | Year 3

C_1 C_2 C_3

- Take treasuries with a succession of maturities and common payment dates
- Bond Price = $D_1C_1 + D_2C_2 + D_3C_3 + 100D_3$

  PV of Cash Flows

  - ✓ Cash flows are known from the bond coupon
  - ✓ The curve has already been built out to 2 years using zeros
  - ✓ D_1 and D_2 are known, calculate D_3 by *bootstrapping*

Problems with Bootstrapping

- Two bonds with same maturity, may have different yields
- More payment dates than bonds
- A good solution is to use regression techniques

The Regression Method

- Regression: $Y = \alpha + \beta X + \varepsilon$
  - ✓ Y is variable you want to predict - e.g. Bond Price
  - ✓ X is "explanatory" variable - e.g. cash flows
  - ✓ β is the multiple to be estimated - discount factor
  - ✓ α is typically insignificant and presumed = 0
  - ✓ ε is error term: ~ iid $No(0, \sigma^2)$
    - Normally distributed
    - Zero mean, constant variance σ^2

Multiple Regression

- Expresses linear relationship between a single dependent variable (y) and a series of independent variables ($x_1 \ldots x_n$)
- $y_i = \alpha + \beta_1 x_{1i} + \beta_2 x_{2i} + \ldots + \beta_n x_{ni} + \varepsilon_i$
- Determine the coefficients which are "optimal":
  - ✓ Least Squares Estimates
    - Coefficients which minimize the sum of squares of the error terms e_i
    - The e_i are the differences between the observed values of y and the values of y estimated using the regression equation

Multiple Regression with Bond Data

- y is the bond price, and (x_1 . . . x_n) are the cash flows on dates 1 to n.
- If α is set to zero, and β_1 . . . β_n can be estimated and will be the discount factors.
- If we make certain statistical assumptions, we can measure how good the estimates are.
- $y_i = \alpha + \beta_1 x_{1i} + \beta_2 x_{2i} + \ldots + \beta_n x_{ni} + \varepsilon_i$
- $\text{Price} = C_1 D_1 + C_2 D_2 + \ldots + C_n D_n + e_i$

Example: Bond Data

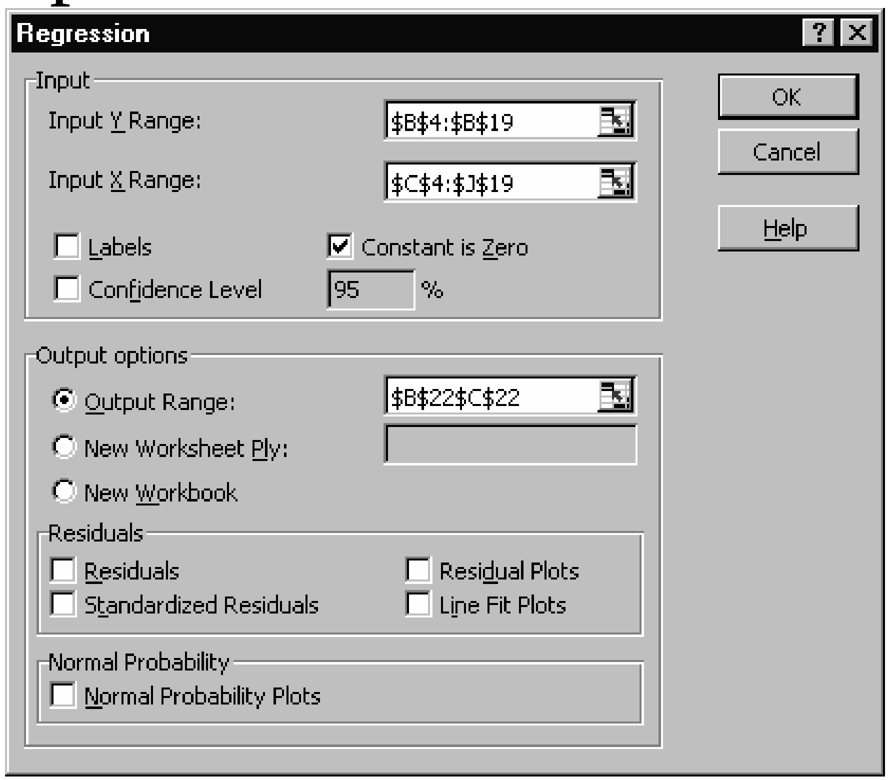
1994	1995	1996	1997	1998	1999	2000	2001	2002
Price				**Cashflow**				
100.91	107.5	0	0	0	0	0	0	0
103.54	110.25	0	0	0	0	0	0	0
101.49	8.5	108.5	0	0	0	0	0	0
102.37	9	109	0	0	0	0	0	0
100.37	8.25	8.25	108.25	0	0	0	0	0
99.74	8	8	108	0	0	0	0	0
99.68	8.25	8.25	8.25	108.25	0	0	0	0
99.70	8.25	8.25	8.25	108.25	0	0	0	0
94.02	7	7	7	7	107	0	0	0
93.02	6.75	6.75	6.75	6.75	106.75	0	0	0
86.88	5.75	5.75	5.75	5.75	5.75	105.75	0	0
86.93	5.75	5.75	5.75	5.75	5.75	105.75	0	0
83.85	5.5	5.5	5.5	5.5	5.5	5.5	105.5	0
87.09	6.13	6.13	6.13	6.13	6.13	6.13	106.13	0
87.58	6.5	6.5	6.5	6.5	6.5	6.5	6.5	106.5
84.78	6	6	6	6	6	6	6	106

Lab: Yield Curve Regression Model

- Worksheet: Data-Regression
- Use Excel Regression Analysis Tool
  - ✓Menu Item: Tools
    - Data Analysis
    - Select Regression
- Estimate the discount factors
- Estimate & plot yield curve
  - ✓Use annual compounding:
    - $R = -1 + (1 / D)^T$

Regression Analysis in Excel

- Select: <Tools>
  - ✓ <Data Analysis> <Regression>
- Fill in the parameters:



Solution: Regression Analysis

Year		*Coefficients*	*Standard Error*	*t Stat*	*Lower 95%*	*Upper 95%*	**Spot Rate**	**Lower 95%**	**Upper 95%**
	Intercept	0	N/A	N/A	N/A	N/A			
1995	D1	0.9389	0.00013	7165.6	0.9386	0.9392	6.50%	6.47%	6.54%
1996	D2	0.8617	0.00013	6547.4	0.8614	0.8620	7.72%	7.70%	7.74%
1997	D3	0.7901	0.00013	5956.9	0.7897	0.7904	8.17%	8.16%	8.19%
1998	D4	0.7235	0.00013	5448.8	0.7232	0.7238	8.43%	8.42%	8.44%
1999	D5	0.6618	0.00013	4926.3	0.6615	0.6622	8.60%	8.59%	8.61%
2000	D6	0.6056	0.00014	4464.8	0.6053	0.6059	8.72%	8.71%	8.73%
2001	D7	0.5559	0.00014	4097.0	0.5556	0.5563	8.75%	8.74%	8.76%
2002	D8	0.5089	0.00014	3758.4	0.5086	0.5092	8.81%	8.80%	8.82%

Estimated Discount Factors

Estimated S.D. of Discount Factors

DF / Std. Error

95% Confidence Intervals: True DF's and Rates will lie between these limits 95%of the time.

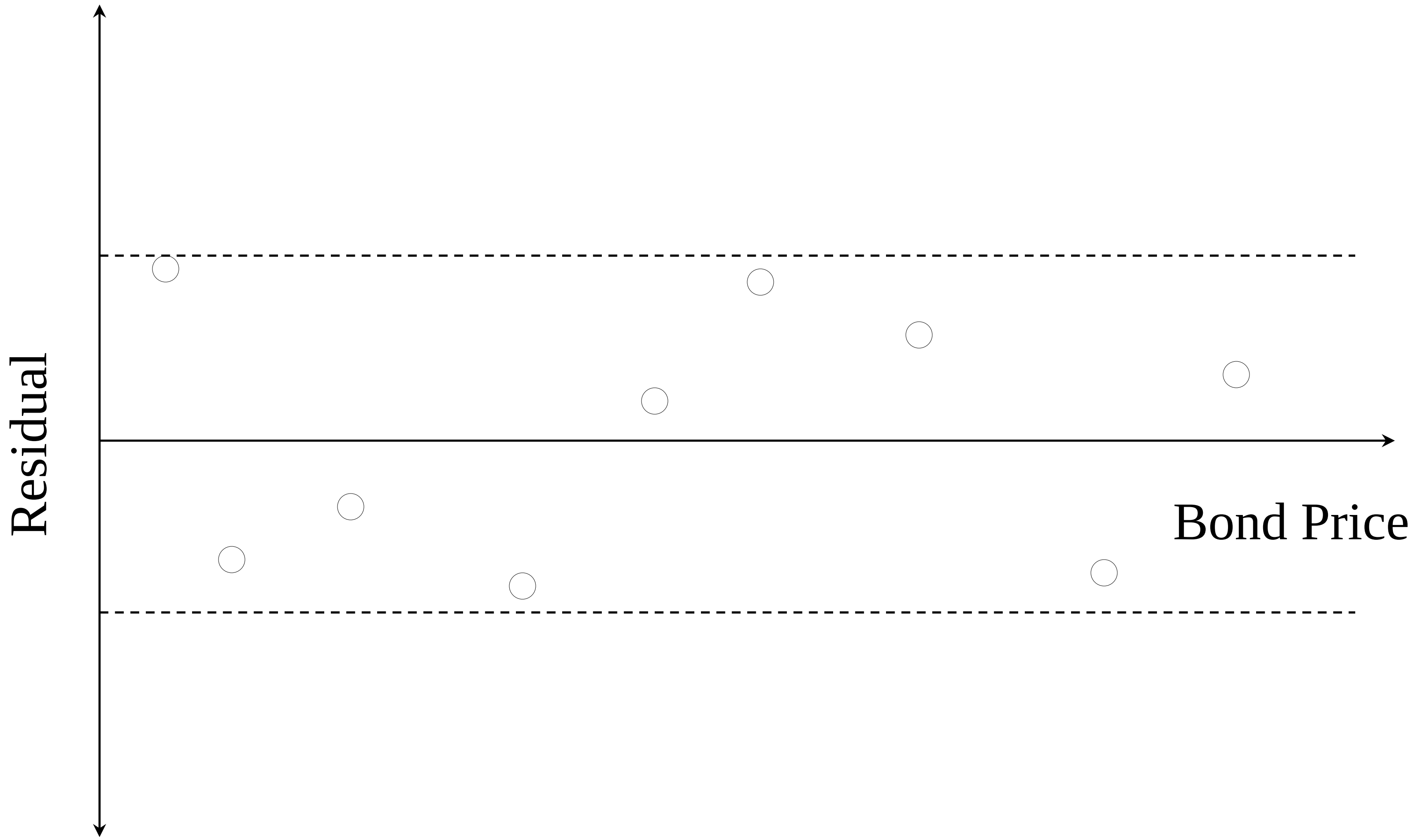
Testing Regression Fit

- R^2 indicates the amount of variance in the dependent variable (price) that is explained by the independent variables (cash flows)
- Partial R^2 indicates the explanatory power of each variable alone
- Standard errors are the square roots of the estimated variances of independent variables
- Confidence intervals are provided by the t and F statistics

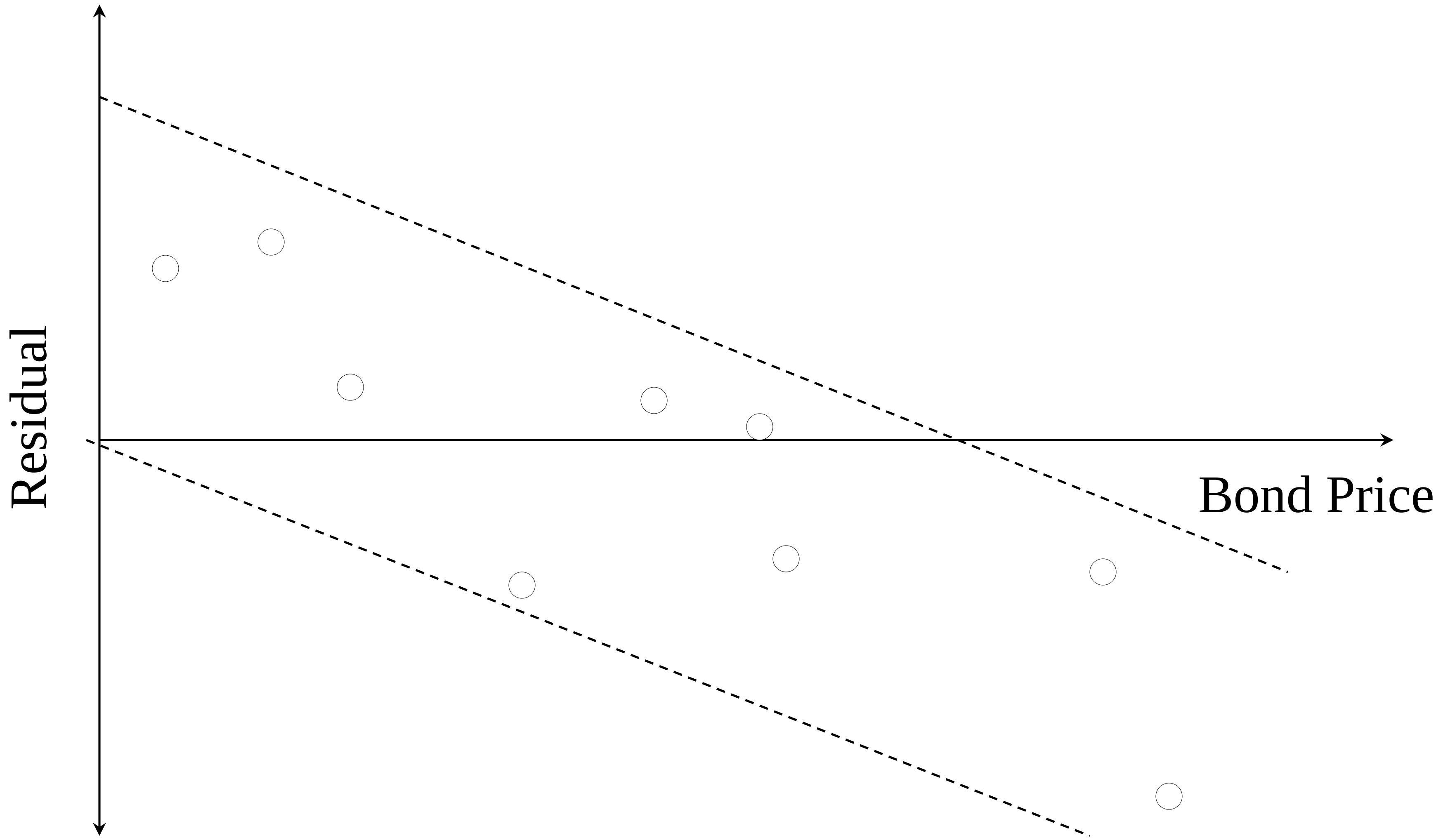
Residuals

- You need to check residuals: $E_i = (Y_i - Y_i^*)$
  - Residual = Actual Price - Predicted Price
- Residual Plot: Residual vs. Bond Price
  - Residual plot should be random scatter around zero
  - If not, it implies poor fit, confidence intervals invalid
  - However, estimates of DF’s are still the best we can achieve, but we can’t say how good they are likely to be.
- Test for:
  - ✓Non-Normality of residuals
  - ✓Bias: non-zero mean
  - ✓Heteroscedasticity (non-constant variance)

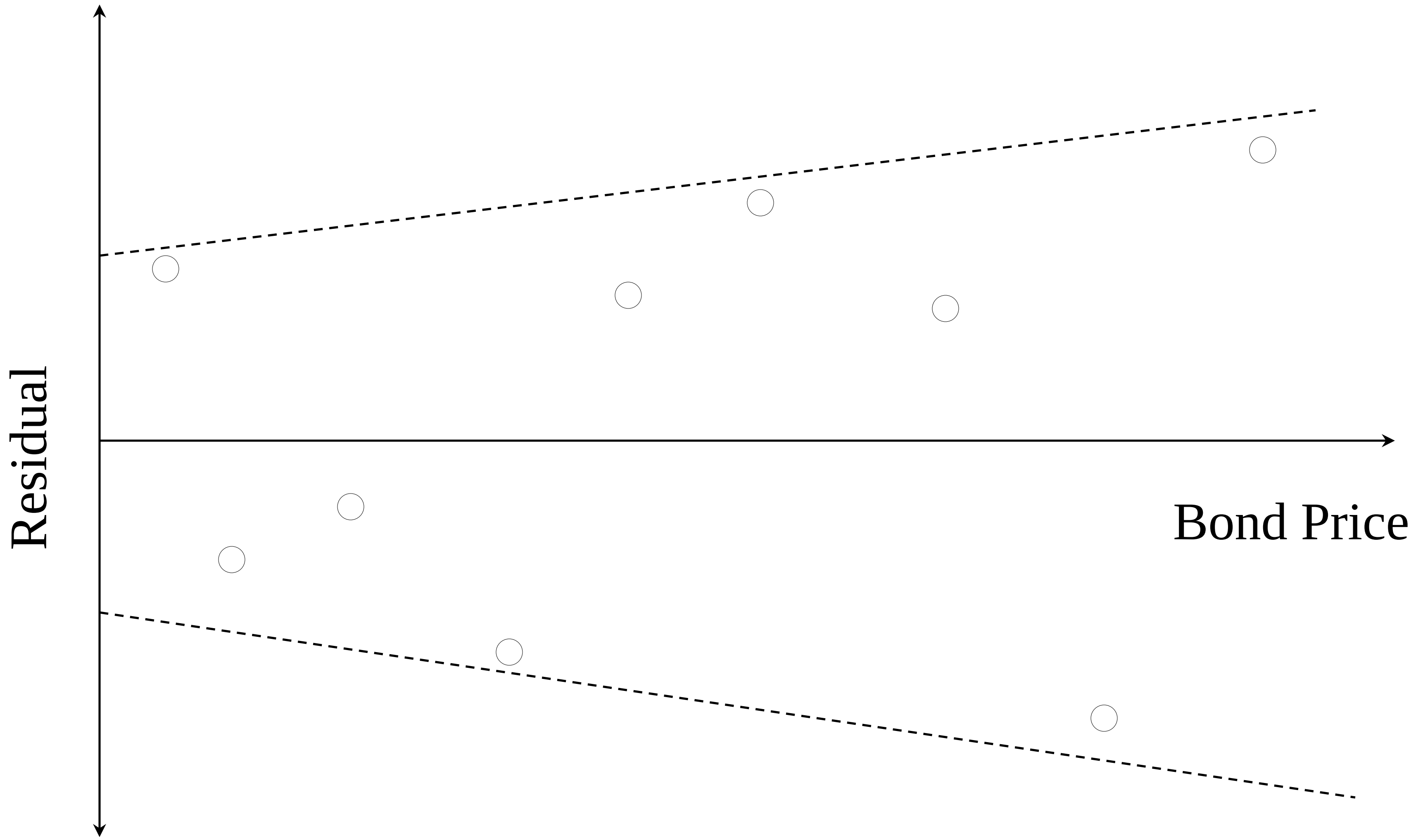
Residual Plot



Residual Plot - Bias



Residual Plot - Heteroscedasticity



Confidence Intervals

- Sample discount factor DF is an *unbiased estimate* of the 'true' discount factor DF_{TRUE}
- Confidence interval: 95% certain that:
  - ✓ $DF_{LOWER} < DF_{TRUE} < DF_{UPPER}$
  - ✓ We can estimate this range from regression model, provided assumptions hold
- Confidence interval for Spot Rate:
  - ✓ $-1+(1/DF_{UPPER})^{(1/t)} < S_t < -1+(1/DF_{LOWER})^{(1/t)}$

Modeling Credit Risk Factors

- Simple regression model:
  - ✓ y is the bond price, and ($x_1 \ldots x_n$) are the cash flows on dates 1 to n.
- $y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \ldots + \beta_n x_{ni} + \varepsilon_i$
- Now additional risk factors r_1, r_2, etc.
  - ✓ E.g. r_1 = country, r_2 = credit rating, etc.
- Model:
  - ✓ $y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \ldots + \beta_n x_{ni} + \alpha_1 r_{1i} + \alpha_2 r_{2i} + \ldots + \alpha_m r_{ni} + \varepsilon_i$

Stepwise Regression

- Forward
  - ✓Start with basic model
  - ✓Add in extra variables one at a time
  - ✓Check goodness of fit, significance of new variable
  - ✓If useful retain, otherwise discard
  - ✓Repeat for other variables
- Backwards
  - ✓Start with full model
  - ✓Eliminate variables one at a time
  - ✓Test fit etc
  - ✓Repeat for other variables

Limitations of Regression Models

- Normal distribution: of error terms for confidence intervals
- Nonlinearity: in relationships causes problems
- Heteroscedasticity: variance is not constant.
- Multicollinearity: Independent variables are correlated
  - ✓ As a group explain the dependent variable well, but the effect of each one can't be estimated properly.

Emerging Market Yield Curves

- Problems:
  - ✓Very few bonds
  - ✓Many are not traded
  - ✓Market very volatile
- Require method which deals with these and which:
  - ✓Fits the data well
  - ✓Produces a smooth curve

Bootstrapping Emerging Market Yield Curves

- Standard method:
  - ✓ Assumes can determine PV of all coupons
  - ✓ Starts with one bond, progresses to next one in maturity order
  - ✓ Usually not enough data to do this.
- Iterative method
  - ✓ Bootstraps all bonds simultaneously
  - ✓ Iterates to a solution

Iterative Method

- Start with first guess of zero curve
- Simultaneously bootstrap all bonds
- Use least squares to get smooth curve
- Use this curve to discount cashflows in the next iteration

Iterative Method Notation

- Notation:
  - ✓ P_k is the price of bond k
  - ✓ C_k is the periodic coupon of bond k
  - ✓ $y_j(t)$ is the j^{th} approximate fit for the zero-coupon curve, starting at $y_1(t)$ as a first guess
  - ✓ $t_i^{(k)}$ is the time to a coupon date for bond P_k
  - ✓ $Z(t_i)$ is the zero-coupon yield as a function of time to maturity t_i
  - ✓ The idea is to iterate from $y_j(t)$ to $Z(t)$

Iterative Method Formulation

$$P_k = \sum_{i=1}^{n_k - 1} c_k e^{-t_i^{(k)} y_j (t_i^{(k)})} + (1 + c_k) e^{-t_n^{(k)} y_j^* (t_n^{(k)})}$$

- Coupons are bootstrapped simultaneously
  - ✓Using $y_j(t)$ for each iteration j
  - ✓For all bonds k = 1, . . . , m
- We back the $y^*_j(t_n^{(k)})$ out of the above equation to serve as an input to the next iteration

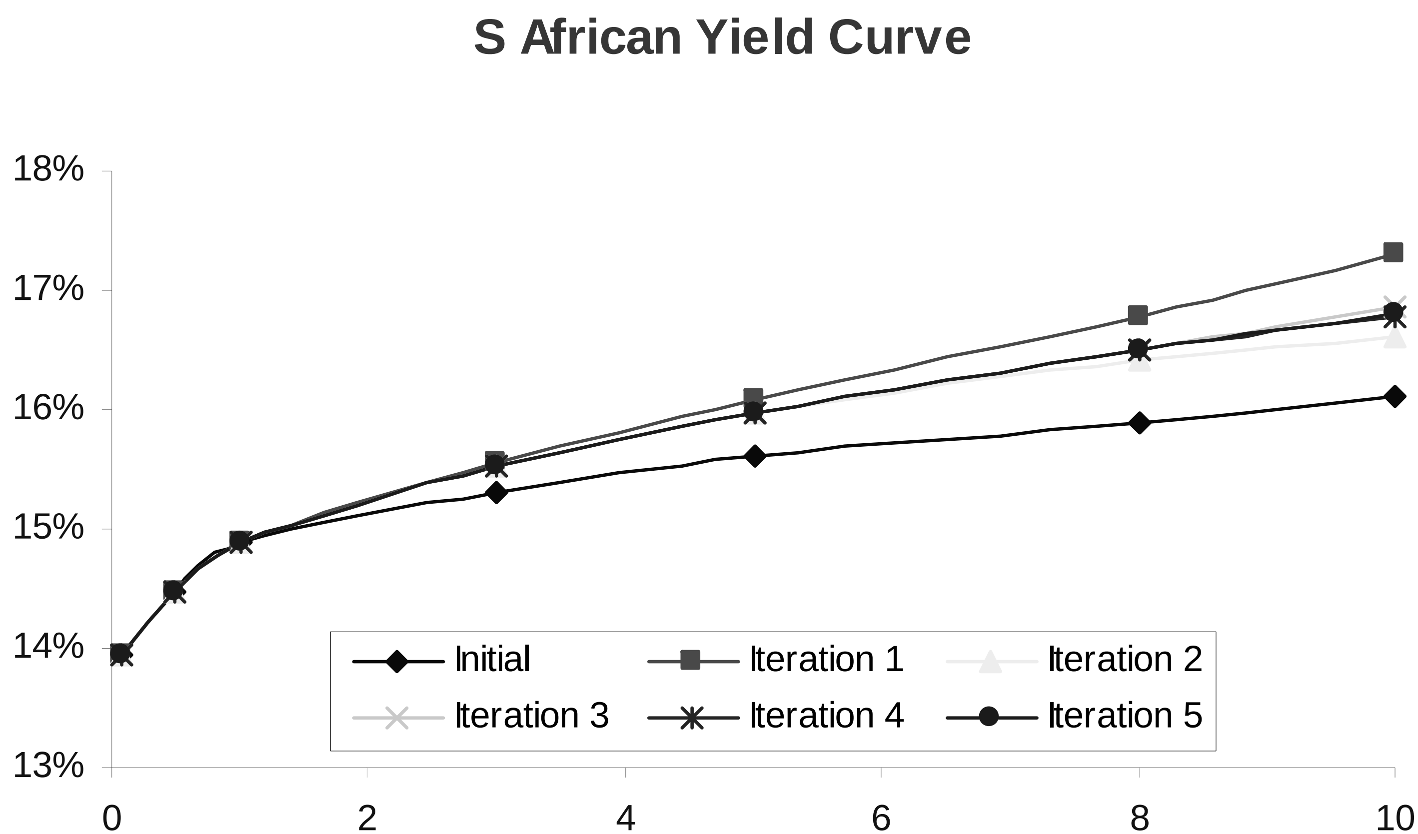
Example - S African Market

- Data
  - ✓Use money market securities out to one year
  - ✓Bonds for the remainder of the curve

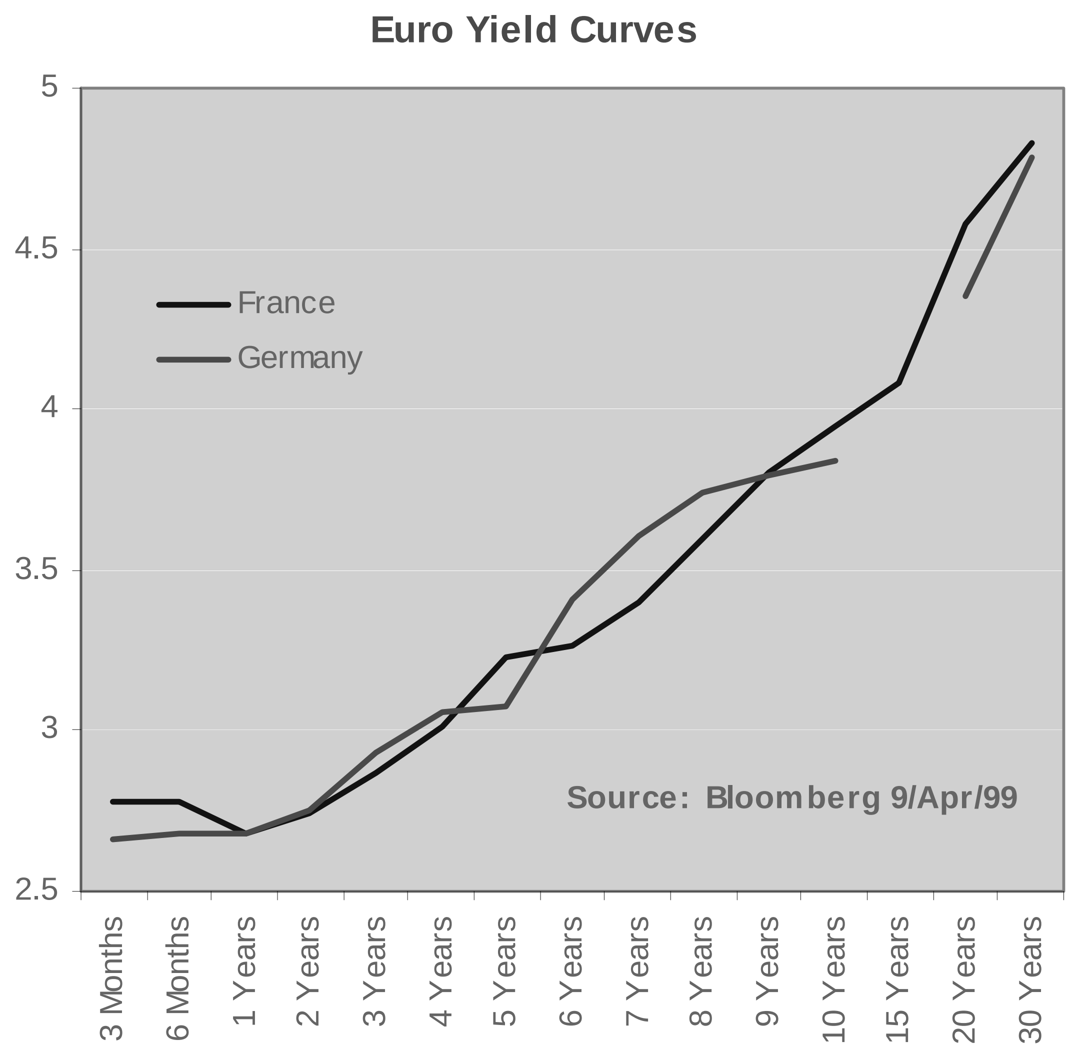
Bond	Maturity	Coupon	Price
1	3	7.50%	97.5110%
2	5	8.00%	98.4596%
3	8	7.00%	87.6629%
4	10	6.50%	80.8032%

- Mo
  - ✓Maturities 1, 6 & 12 month
  - ✓Yields 13.95%, 14.48% and 14.88% respectively

Solution - Yield Curve Construction by Iterative Bootstrap



Euro-Yield Curves



Euro-Yield Curves

- No single, standard curve
  - ✓Anomolies and strange spread differentials
    - 7 Year: Bund cheaper than OAT by 20 bp
    - 20 Year: OAT 23 bp cheaper than Bund
- No natural spread France vs. Germany
  - ✓Short End: OATs and BTNs more liquid than Bunds
    - French repo market more efficient
  - ✓Old OATs of 2008 bought up by insurance companies due to tax benefits on 8-year contracts
- Several Gvts. competing for benchmark status
    - Issuers sometimes price off Bunds, OATs or both!
    - Most traders use swap curve to price bonds

Summary: Yield Curve Building with Bonds

- Bootstrap Method
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- Building Emerging Market Yield Curves
- Iterative Methods