Interpolation Techniques

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- Why interpolate?
- Straight line interpolation
- Cubic spline interpolation
- Basis spline interpolation

Why Interpolate

- Structuring
  - Project security cash flows
  - Need forward rates on coupon dates
- Valuation
  - Need spot rates on coupon dates
- In either case coupon dates may not coincide with dates for which zero-coupon yields are known.

Interpolation Methods

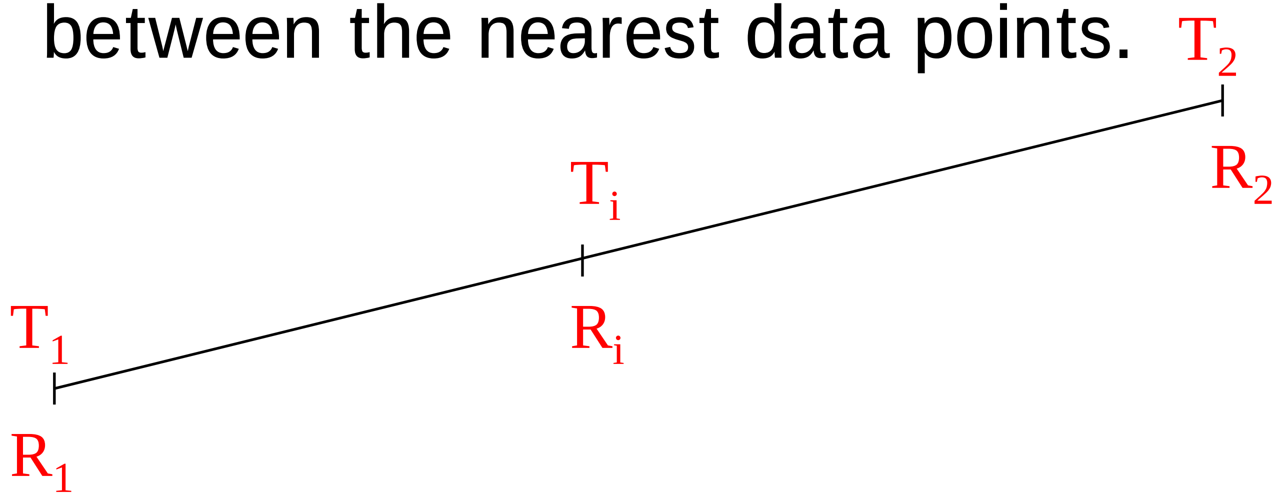
- Straight Line
- Polynomial
  - Single high order polynomial
  - Unstable between points and at ends
- Splined polynomial
  - Low order polynomials linked together
- Basis Splines
  - Represent discount function as weighted sum of other functions

Straight Line Interpolation – Pros and Cons

- Simple to estimate intermediate points on curve
- Not accurate for undulating curves
- Gives different results on discount factors
- Produces discontinuous forward rate curve

Linear Interpolation

- Intermediate values lie on a straight line between the nearest data points.



- $R_i = R_1 + (R_2 - R_1) \text{ x } (T_i - T_1) / (T_2 - T_1)$

Linear Interpolation: Rates or Discount Factors?

- If interest rates lie on a straight line, discount factors do not
- Example:
  - Using Rates
    R_1 = R2 = 5.00%
    T_1 = 90
    T_2 = 180
    T_i = 120
    R_i = 5.00%

    Using DF's
    D_1 = 0.9877
    D_2 = 0.9756
    D_i = 0.9836

    R_i = 4.99%

Linear and Exponential Interpolation

- Linear interpolation on continuously compounded interest rates is equivalent to exponential interpolation on discount factors

$$D_1 = e^{-R_1 T_1}, D2 = e^{-R_2 T_2}$$

$$R_i = (1-\alpha) R_1 + \alpha R_2$$

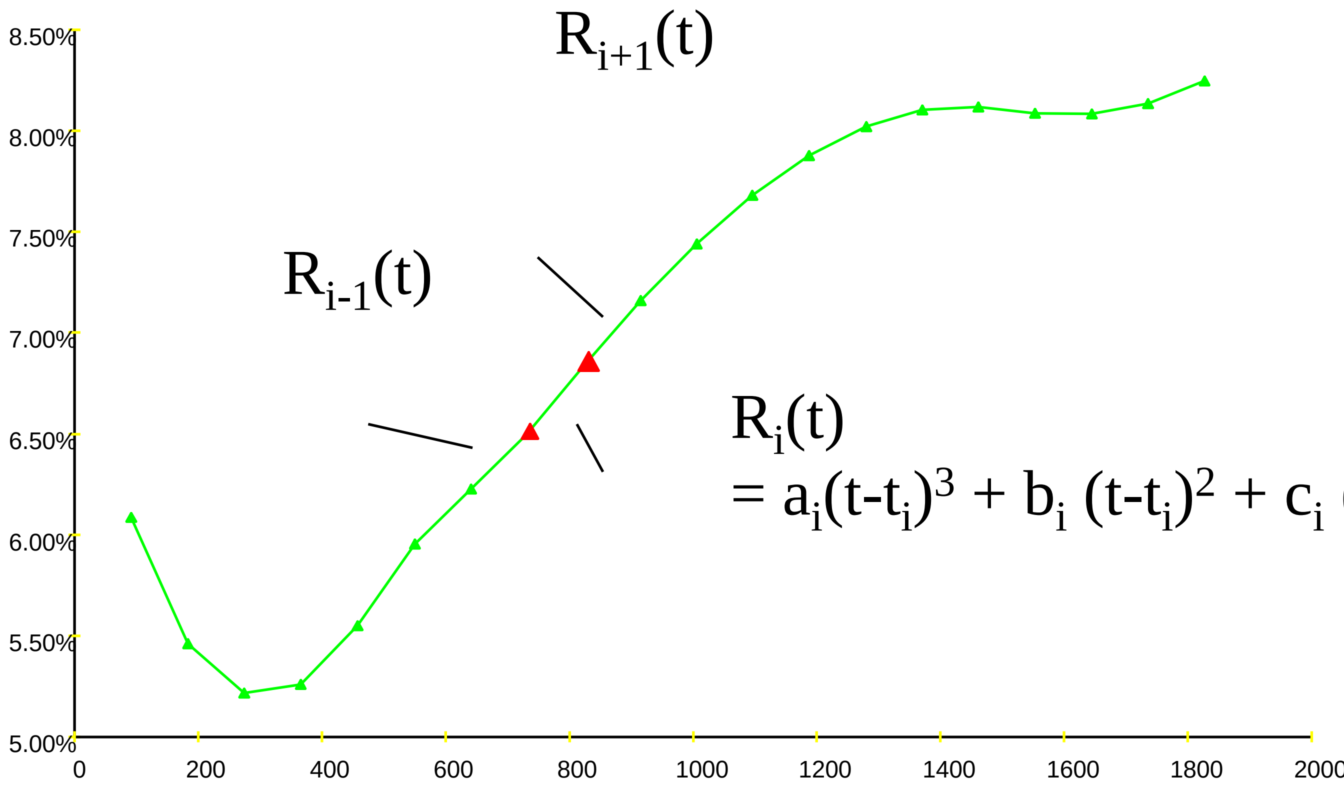
$$\alpha = \frac{T_i - T_1}{T_2 - T_1}$$

$$\Rightarrow D_i = D_1^{(1-\alpha)\frac{T_i}{T_1}} D_2^{\alpha \frac{T_i}{T_2}}$$

Cubic Spline Interpolation

- A different cubic polynomial is fitted between each pair of data points
- The polynomials are twice differentiable
- Ensures that:
  - The slope of the curve is smooth
  - The rate of change of the slope is smooth
  - The curves “join” at the end points

Cubic Spline Curve Fitting



$R_i(t)$

$= a_i(t-t_i)^3 + b_i\,(t-t_i)^2 + c_i\,(t-t_i) + d_i$

Natural Splines

- End Curve Conditions
  - Conditions of the two ends of the yield curve must be specified for a solution.
- Natural Spline
  - Second derivative (rate of change of the slope of the yield curve) equal to zero at both ends.
    - Slope of curve is constant at the ends
  - You typically only care about points in the belly of the curve

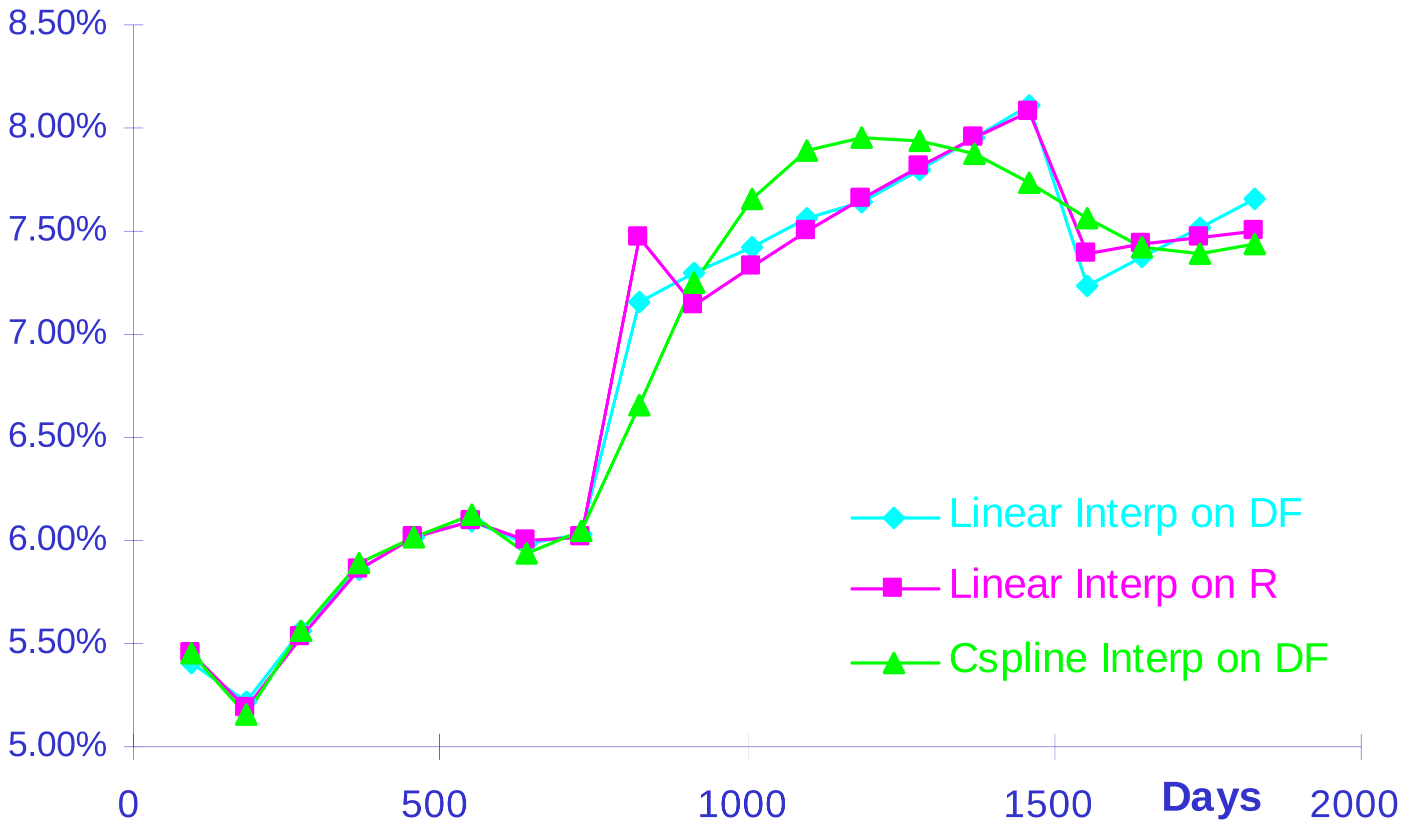
Cubic Splines – Pros & Cons

- Smooth curve -twice differentiable at every data point
- Can be used on both rates and DF's
- Works for undulating curves
- Produces continuous forward rate curve
- Not so easy to calculate
- Can suffer from oscillation

Lab: Building Yield Curves with Cubic Splines

- Excel workbook; Yield Curve Modeling.xls
- Worksheet: Cubic Spline Curve
- Build 3m forward rate curve using:
  - Linearly interpolated DFs
  - Linearly interpolated spot rates
  - Cubic Spline interpolated spot rates
- See Notes & Solution

Solution: Cubic Spline Forward Curves



Basis Splines

- Another widely used interpolation method
- Used for modeling *discount function*
- Typically combined with regression analysis

Regression: More Payment Dates than Bonds

- This is the usual case, as bond coupon dates fall on different days in the year.
- Have to represent discount factors by a function
  - Insufficient bonds to estimate model parameters
    - Singular matrix
- Use regression to determine parameters of the discount function
  - Then calculate discount factors on any chosen date

Representing the Discount Function by Basis Splines

- Represent DF's by function d(t):

$$d(t) = \sum_{l=1}^{L} \alpha_l f_l(t)$$

l = 1...L: the number of basis spline functions f.

α: weights applied to each function

- Bond prices can be expressed as the sum of discounted cash flows:

$$P_i = \sum_{j=1}^{n} C_{ij} \sum_{l=1}^{L} \alpha_l f_l(t)$$

C: Bond cash flows

P: Bond price

- Determine values of weights to fit bond prices to market data.

Estimating the Discount Function

- Rearrange bond price equation:
  - Sum of discounted cash flows

$$P_i = \sum_{j=1}^{n} C_{ij} \sum_{l=1}^{L} \alpha_l f_l(t)$$

  - Sum of weighted cashflow x spline function

$$P_i = \sum_{l=1}^{L} \alpha_l \sum_{j=1}^{n} C_{ij} f_l(t)$$

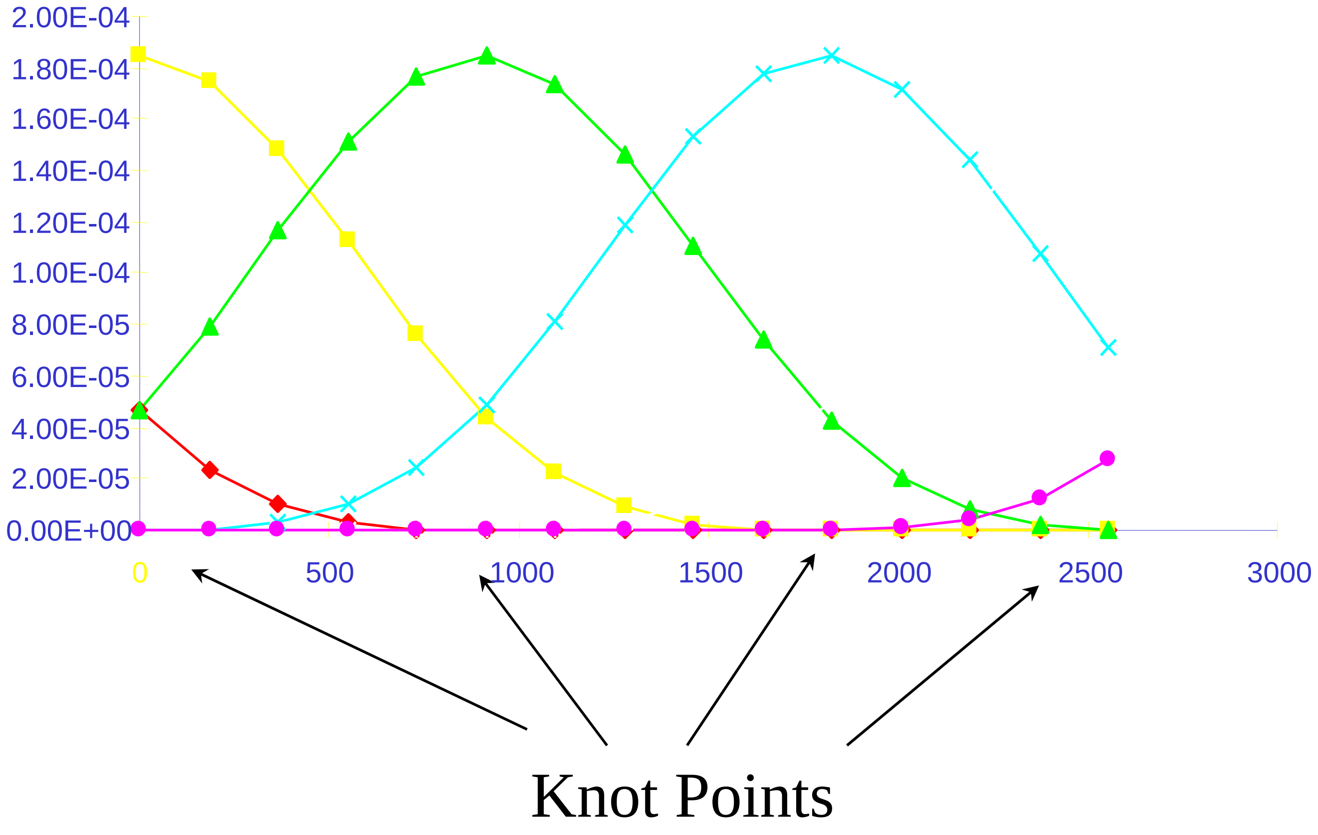
- Spline functions are defined over the whole period
- As long as we have more bonds than weighting factors α, regression can be used.

Basis Splines & Knot Points

- Basis Splines
  - The discount function is a weighted average of a number of overlapping B-Splines.
  - Cubic B-Spline functions usually selected.
  - Individual spline functions are not linked.
- Knot Points
  - Each spline function is non-zero over a well-defined interval.
  - The start and end points of the splines are called “knot points”.

Basis Spline Curves

- The discount function is the weighted sum of individual splines



Selection of Knot Points

- Results can be sensitive to placing of knot points
  - Unless there is an even distribution of bonds.
- Important to have an equal number of bonds with maturities between each knot point.
  - Reduces estimation error.

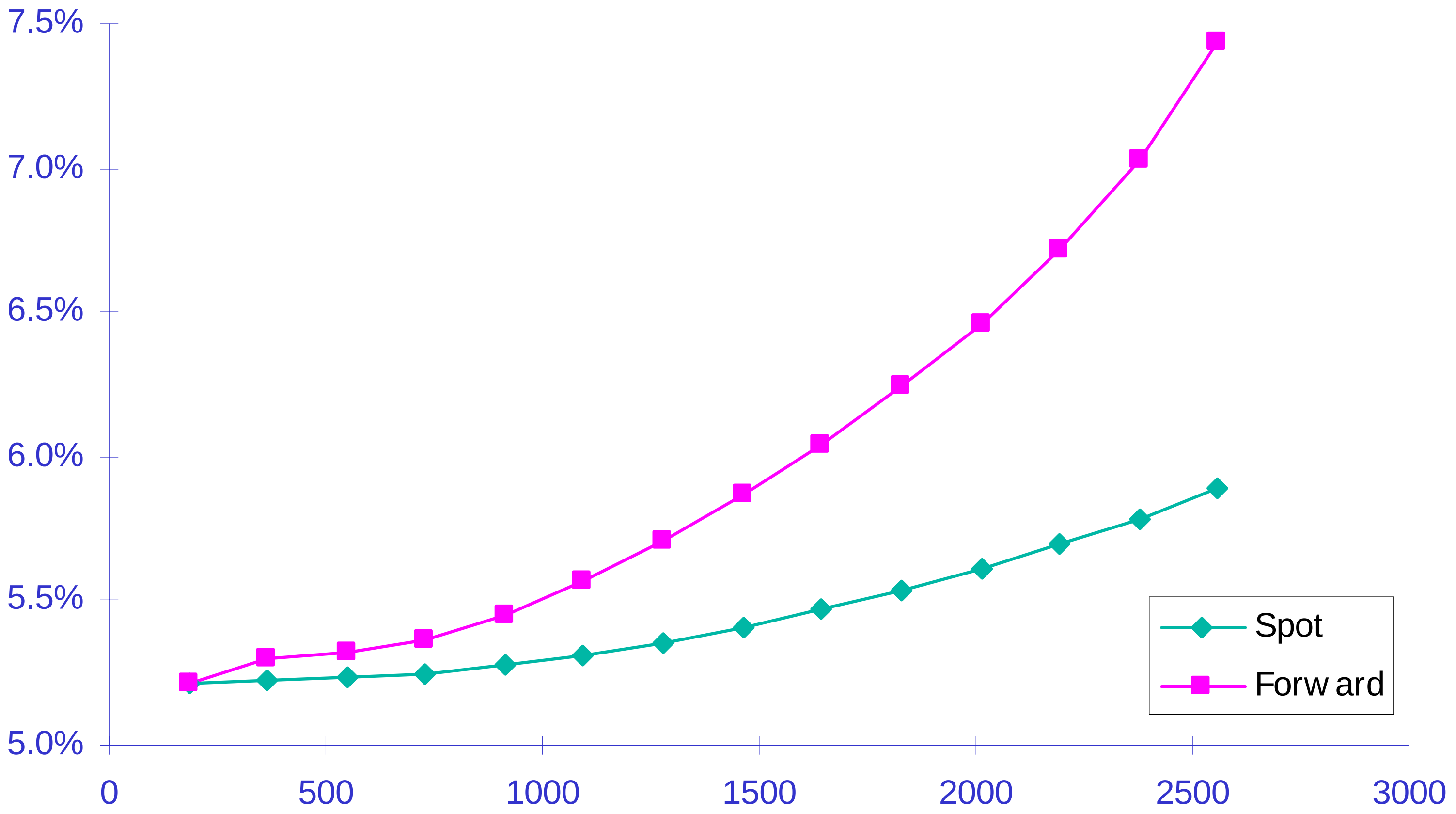
Building a Zero Coupon Curve from Treasury Bonds

- Often have more payment dates than bonds
  - No unique set of discount factors that will price all bonds
- Use Regression Analysis
  - Determine Least Squares Estimates of Discount Factors
    - Minimize the square of the difference between the observed bond prices and those based on estimated discount factors.
- Discount factors must be linked by a functional form
  - Cubic splines have problems due to correlation.
  - Basis splines are independent but watch “knot points”.

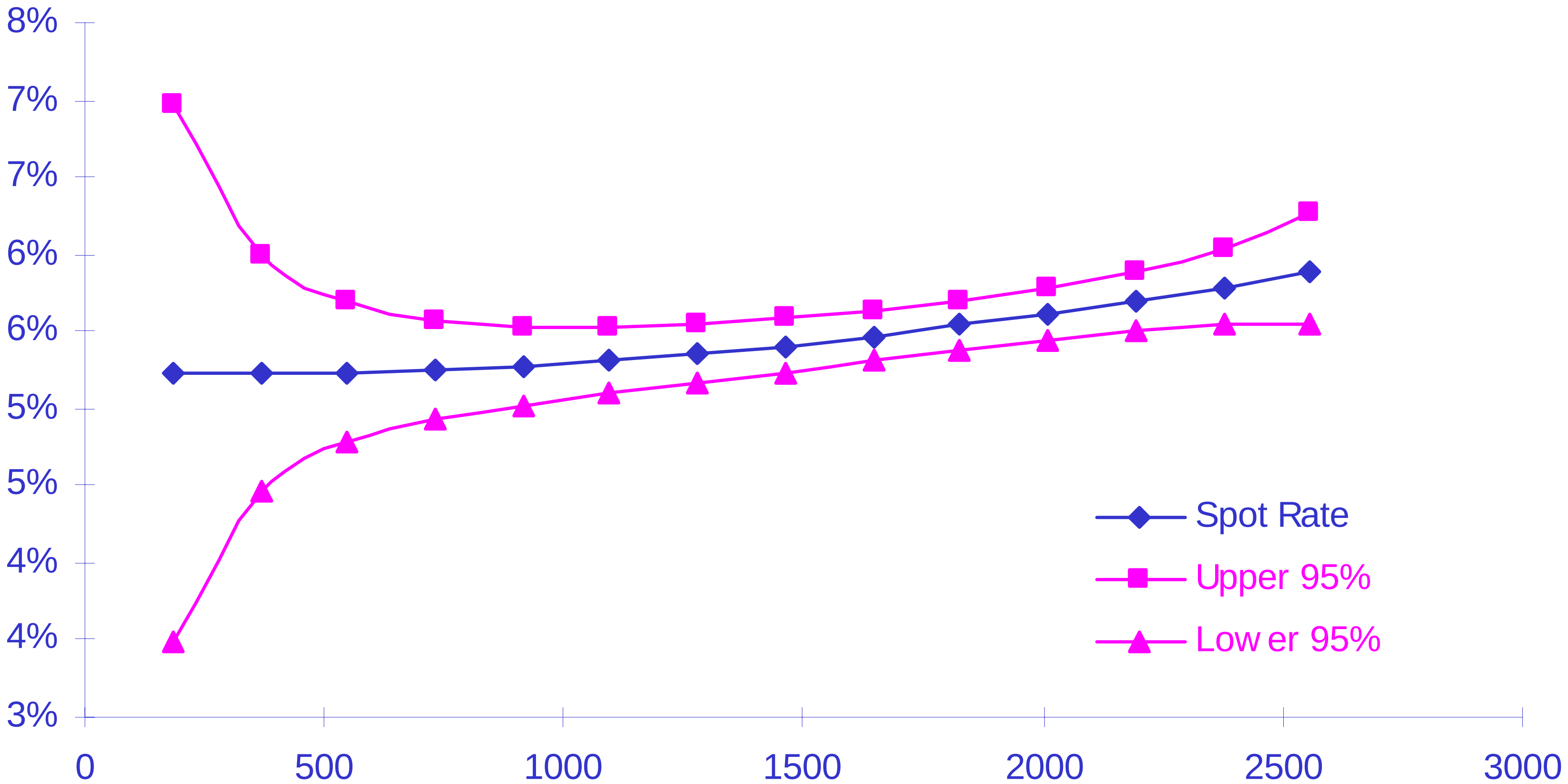
Lab: Building a Yield Curve with Basis Splines

- Worksheet: Basis Splines
- Build yield curve using bond data
- Method:
  - Basis Splines & Regression
- See Notes & Solution

Spot and Forward Rate Curves



Confidence Intervals



Interpolation Methods: Summary

- Straight Line Interpolation
  - Inaccurate.
  - Leads to discontinuous forward rates.
- Cubic Splines
  - Better than linear interpolation.
  - Due to smoothness condition points on the yield curve are linked together.
  - Linking causes multicollinearity.
  - Accuracy of and one discount factor cannot be determined.
- Basis Splines
  - Functions go to zero at defined points.
  - Need to use a weighted combination of several B-Splines.