Interest Rate Models

Interest Rate Models

- Model types, characteristics
- Model Taxonomy
- One-Factor models
  - Vasicek
  - Ho & Lee
  - Hull & White
  - Black-Derman-Toy Model

Two Factor Models

  - Fong & Vasicek
  - Longstaff & Scwartz
  - Hull & White
  - Heath-Jarrow-Morton

Interest Rate Models

- Used to:
  - Value derivatives, esp. non-standard
  - Compute hedge ratios
  - Assess portfolio risk
- Provides a consistent framework for valuation, hedging & risk-management

Types of Model

- Extensions of Black-Scholes
  - Widely used for caps/floors (Black's model)
- Models of the short term interest rate
  - Easy to implement
  - Many varieties, e.g. BDT, HW
- Models of entire yield curve
  - Most difficult
  - Usually simplified to two factors (e.g. HJM)

Caps, Floors & Collars

- Very popular instruments
  - Great demand for caps due to increased interest rate volatility
  - Market very liquid
  - Used to calculate market's view of interest rate volatility

Caps, Floors & Collars

- Caps:
  - Limits Upside Risk / Gain
    - Series of interest rate call options
    - Caps interest rate, or equity index return
- Floor:
  - Limits Downside Risk / Gain
    - Series of interest rate put options
- Collar
  - Combines Cap & Floor
  - Fixes interest rate or equity index within a band

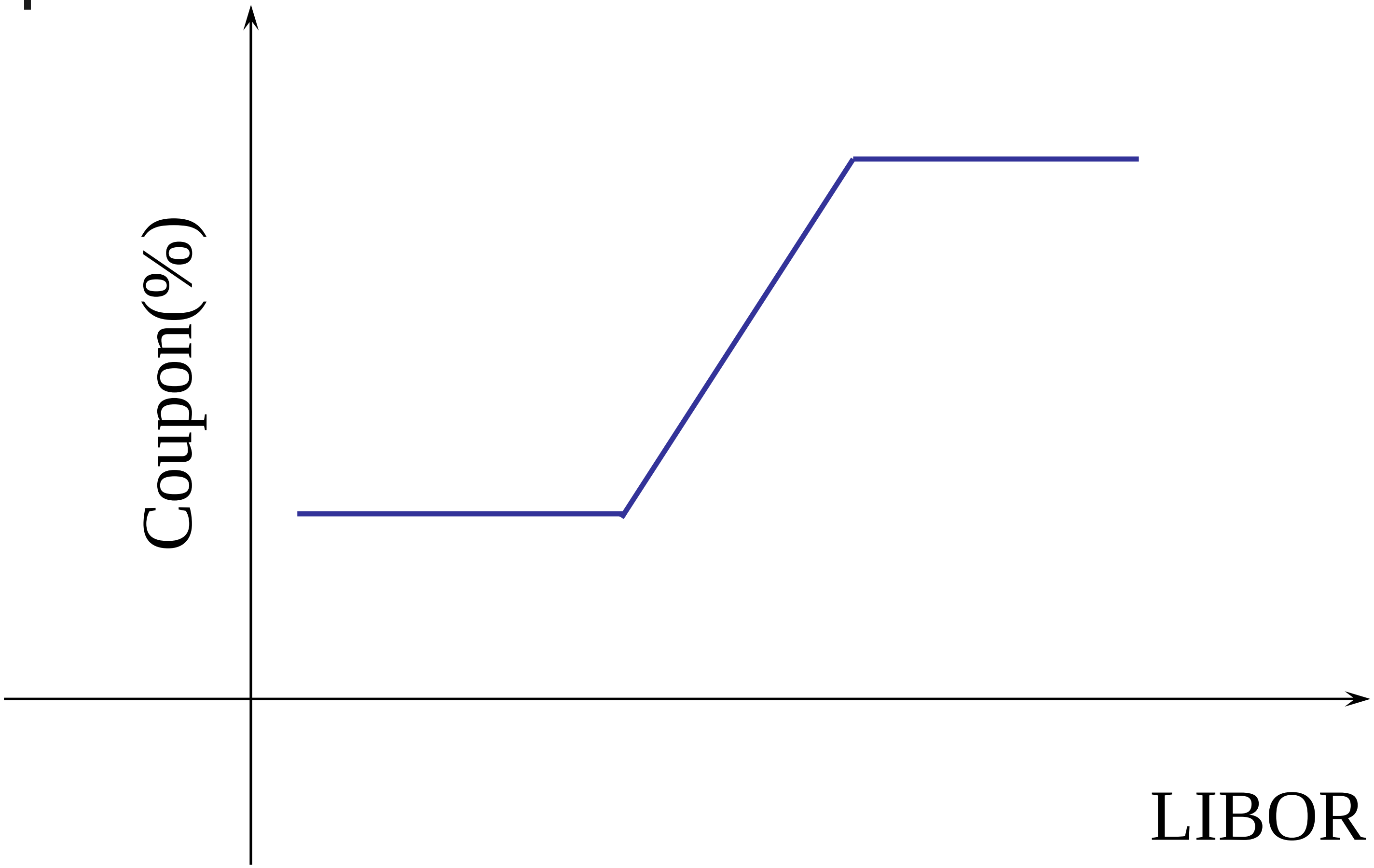
Interest Rate Caps

- Contract terms
  - Cap strike rate, R_x (7%)
  - Term (3 years)
  - Reset frequency (quarterly)
  - Reference rate (LIBOR)
  - Principal ($1MM)
- Payment from seller to buyer:
  - 0.25 x $1MM x Max(LIBOR - R_x, 0)
    - In arrears, usually starts after 3 months
    - Each piece is called a “caplet”

Collar

- Combines Floor(s) and Cap(s)
  - Limits upside potential and downside risk
  - Sale of call(s) & purchase of put(s)
  - Premium from calls offsets cost of puts
- Zero Cost Collar:
  - Special case where Put Premium = Call Premium
  - Net cost is zero
- Typically used to lock in gains after market rally

Collared FRN



Black's Model

- Simple extension of Black-Scholes
  - Originally developed for commodity futures
  - Used to value caps and floors
  - Let F = forward price, X = strike price
  - Value of call option:

$$C = e^{-rt}[FN(d_1) - XN(d_2)]$$

$$d_1 = \frac{\ln(F/X) + (\sigma^2/2)t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

Application to Caps

- Example: 1-year cap
  - NP = notional principal
  - R_j = reference rate at reset period j
  - R_x = strike rate
  - Then, get NP x Max$\{R_j - R_x, 0\}$ in arrears
  - But this is an option on R_j, not F_j
  - Use F_j as an estimator of R_j and apply Black's model to F_j
    - Previously was a forward price, now a forward rate

Black's Model for Caps

- Payments: NP x Max{R_j - R_x,0} in arrears
- These are a series of options:
  - One for each R_j , the future spot interest rate
  - Called *caplets*
- Let F_j = forward rate from j to j+1
- Value of caplet j:
  - Discount by (1+ F_j) as paid in arrears

$$C = NP \times e^{-rt}[F_j N(d_1) - R_x N(d_2)] / (1 + F_j)$$

Black's Model - Example

- 8% cap on 3-m LIBOR (R_x = Strike = 8%)
  - Capped for period of 3m, in 1-year's time
  - f = 1-year forward rate for 3m LIBOR is 7%
  - R_f = 1-year spot rate is 6.5%
  - Yield volatility is 20% pa
- See Excel workbook Swaps.xls
  - Black's Model - Example Spreadsheet

Black's Model - Example

Strike Term Fwd Rate Vol R_f C/P E/A HCost

- C = BSOpt (8%, 1, 0, 7%, 20%, 6.5%, 0, 0, 0)
  - Holding Cost
    - Hcost = (R_f-d) for stocks, 0 for Futures
  - Cap Premium = 0.00211
- Convert to %: C% = C x t / (1 + F * t)
  - 0.00211 x 0.25 x 1 / (1 + 7% x 0.25)
  - Cap Premium % = 0.0518% (5.18bp)
  - So cost of capping $1000,000 loan would be $518

Black's Model - Equivalent Formulation in Terms of Price

- Cap = *Put option on price*
  - Equivalent of *call option on rate*
    - Useful if know price volatility rather than yield vol.
- F = 1 / (1 + f x t) is *forward price*
  - F = 1 / (1 + 7% x 0.25) = 0.982801
- X = 1/(1 + R_X x t) is *strike price*
  - X = 1 / (1 + 8% x 0.25) = 0.980392
- Require *price volatility*
  - Other parameters as before

Black's Model - Price Example

Strike Term Fwd Price Vol R_f C/P E/A

- C = BSOpt (.980392, 1, 0, 0.982801, 0.3702%, 6.5%, 1, 0, 0)

HCost

Cap Premium % = 0.0518% (5.18bp)

- So cost of capping $1000,000 loan would be $518

- NOTE:
  - Premium already expressed as % of FV
  - This time we are price a *put* option

Lab: Cap, Floor & Collar pricing - Black's Model

- Excel Workbook, Swaps.xls
  - Black's Model - Worksheet
    - See lab writeup, written solution & solution spreadsheet
- Pricing a 1 year cap on 3-m LIBOR
  - Quarterly resets, so 4 caplets
  - Given price volatility, so use price formulation
  - Back out forward rates from spot rates

Solution: Cap, Floor & Collar Pricing - Black's Model

CAP	03-Aug-96	15-Oct-96	15-Jan-97	15-Apr-97	15-Jul-97
Strike Rate		7.500%	7.500%	7.500%	7.500%
Strike Price		0.9850	0.9812	0.9816	0.9814
Caplet Cost		0.0299%	0.0686%	0.1531%	0.2190%
Cap Cost		0.470497%			
FLOOR	03-Aug-96	**15-Oct-96**	**15-Jan-97**	**15-Apr-97**	**15-Jul-97**
Strike Rate		4.736%	4.736%	4.736%	4.736%
Strike Price		0.9905	0.9880	0.9883	0.9882
Floorlet Cost		0.083%	0.123%	0.126%	0.138%
Floor Cost		0.470497%			

Limitations of Black's Model

- Problems:
  - Unbiasedness: empirically false
    - Option on R_j not same as option on F_j
  - Discount rate: fixed - but F_j variable
    - Rates both stochastic and fixed!
  - If applied to prices the additional problem
    - Assumes prices can be any positive number
    - But can't exceed value of future cash flows

Swaptions

- Option on a swap
  - Right to enter a swap at known fixed rate
    - *Receiver Swaption*: right to receive fixed
    - *Payer Swaption*: right to pay fixed
  - Essentially a bond option with strike = notional
    - When exercised, will exchange floating for fixed
    - Fixed payments correspond to a bond

Swaptions: Applications

- Anticipatory financing
  - manage future borrowing cost
    - e.g., bidding for a contract; if get it, will want to swap, lock in today's rates
    - e.g., option to build a plant, so buy swaption
- Change terms of existing swap
    - Cancelable, putable swap

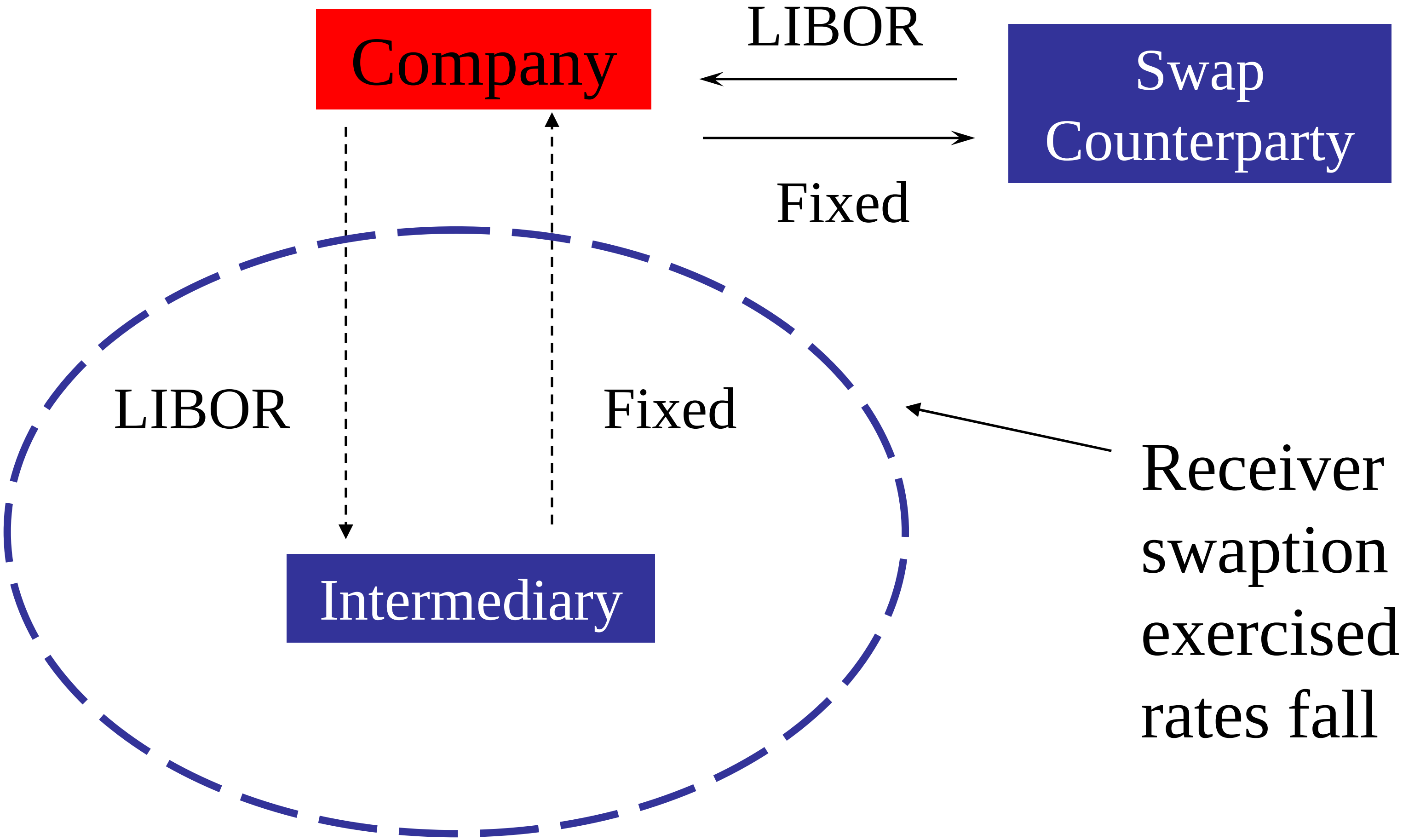
Example Swaption Strategies

- Floating rate borrower doesn't believe rates will fall. How can he reduce funding cost?
  - Sell floor = sell receiver swaption
- Borrower wants to delay decision to lock in rates at 9% for 1 year
  - Buys payer swaption
- Speculator believes rates will rise next year
  - Buy payer swaptions
  - Sell receiver swaptions

Swaptions: Creating Swap Variants

- Extendible Swap:
  - Fixed payer can extend life of swap
    - = payer swap + payer swaption
    - e.g. uncertain about term of financing
- Putable Swap: can cancel swap
    - = payer swap + receiver swaption
    - e.g. financing need disappears
- Cancelable Swap: counterparty can cancel
    - = payer swap - receiver swaption
    - e.g. credit rating worsens

Putable Swap



Receiver swaption - exercised if rates fall

Swaptions & Asset Swaps

- Another application: callable bonds
- Asset swap: issue fixed-rate bond, swap to floating
- If issuer calls bond (most corporates are callable), he is left with swap
- Swaption: allows swap to be cancelled
  - Payer swaption

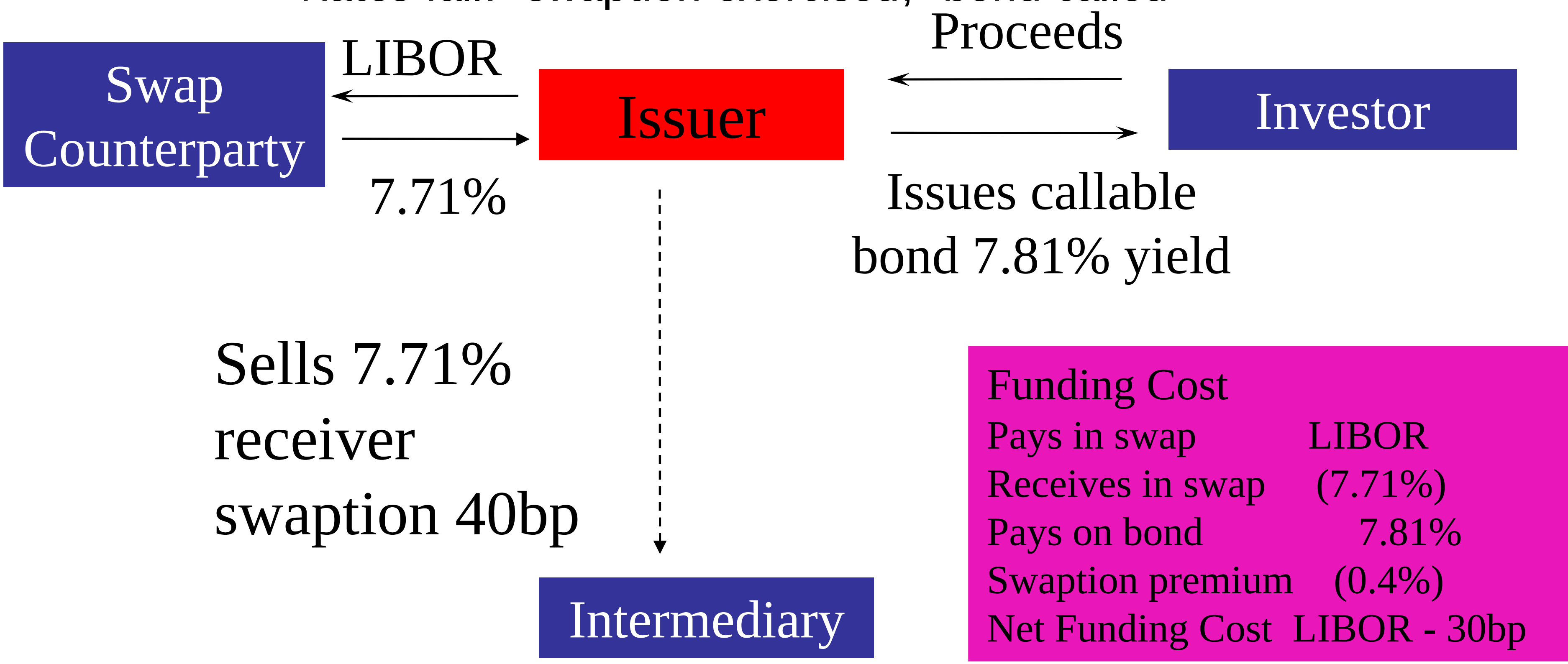
Swaption Arbitrage with Callable Bonds

- Strong demand for receiver swaptions
  - From swap buyers (paying fixed), concerned about rates falling
- Supply: from issuers of callable bonds
- Arbitrage
  - Issuer sells receiver swaption
  - Swaption premium > extra yield on callable bond
    - Nb match term of call provision to term of swaption
      - e.g. callable after 2 years, sell 2-year European swaption

Swaption Arbitrage Example

Action

- Rates rise: no action; swaption not exercised
- Rates fall: swaption exercised; bond called



Black's Model for Swaptions

- Widely used for European Swaptions

$$S = e^{-rT}\left[R_s N(d_1) - R_X N(d_2)\right]$$

$$d_1 = \frac{\left[\ln(R_s / R_X) + \frac{\sigma^2}{2}T\right]}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

  - T is the swaption maturity date
  - R_s is the forward swap rate
  - R_x is the strike

Swaption Example

- Two year swaption on 1 year semi-annual payer swap
  - Strike rate is 6%
  - Forward swap rate volatility is 20%

Maturity	Spot Rate	Spot DF	Forward DF
2.0	5.00%	0.9048	
2.5	5.25%	0.8770	0.9692
3.0	5.50%	0.8479	0.9371

Swaption Example

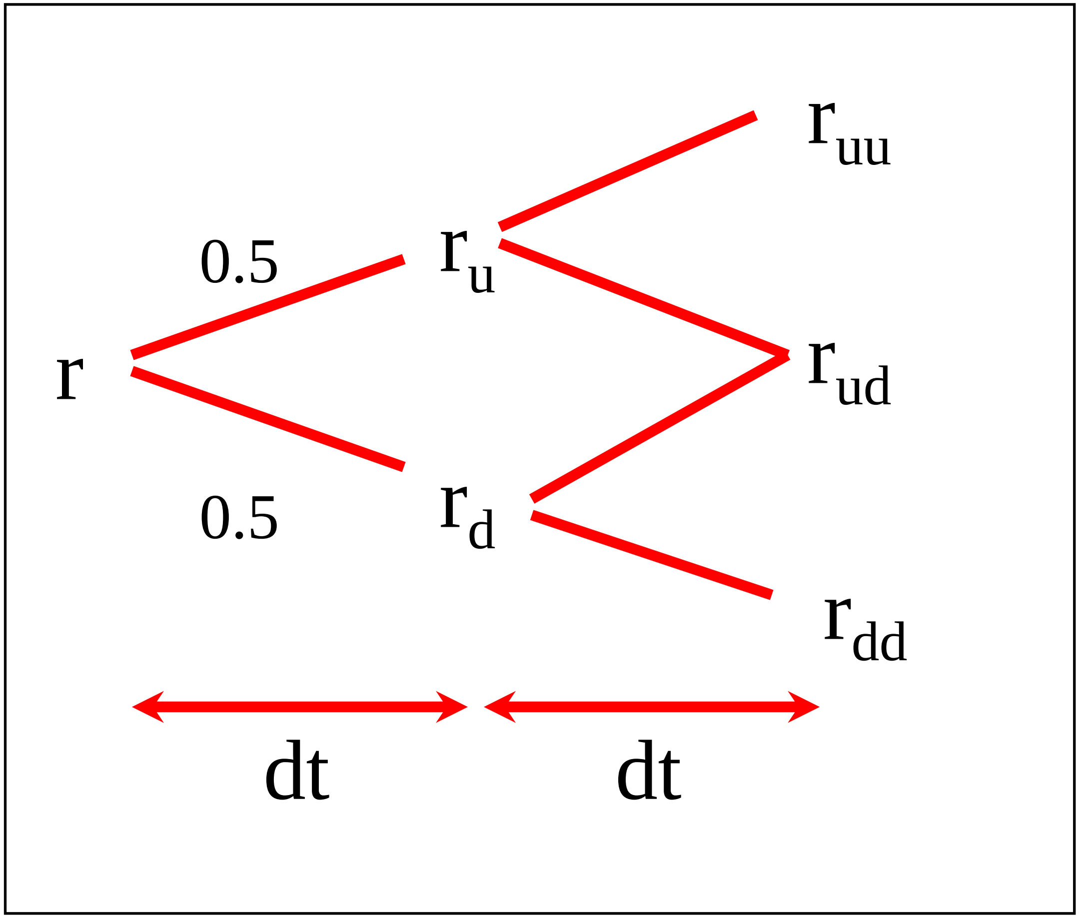
- Forward swap rate R_s
  - $1 = \Sigma R_s DF(T,t_i) + DF(T, t_2)$
    - $DF(T, t_i)$ is forward discount factor from start of swap (maturity of swaption, T = 2) to coupon dates t_i
  - Hence $R_s = 2[1- DF(T, t_2)] / \Sigma R_s DF(T,t_i)$
    - Rs = 2(1-0.9731) / (0.9692 + 0.9371)=6.6%

Strike | Term | Fwd Swap Rate | Vol | R_f | C/P E/A | HCost

- S = BSOpt (6%, 1, 0, 6.6%, 20%, 5%, 0, 0, 0)
- S = 0.95%

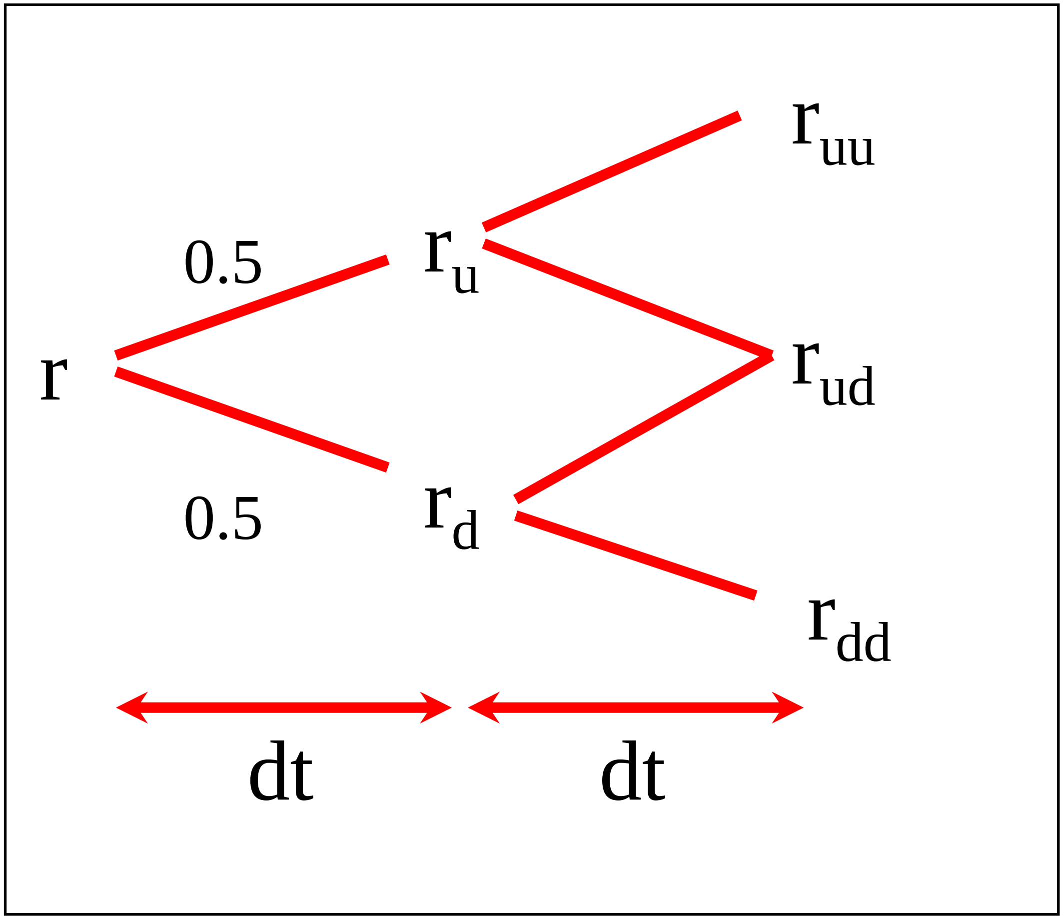
Short-Rate Models

- Specify a lattice of interest rates
- E.g. 3 years, quarterly.
- Then dt = 3m, N = 12
- Risk neutral probability: 1/2



Short-Rate Models

- Here r is a future short term interest rate
- NOT a forward interest rate
- Lattice models the evolution of short-term interest rate



Model Characteristics

- Equilibrium Models
  - Start with economic assumptions
  - Derive short rate process
  - Current yield curve is an *output*
  - Fit to observed yield curve is only approximate
- No Arbitrage Models
  - Designed to be consistent with current term structure
  - Current term structure is an *input*
  - Typically consistent with volatility term structure also (to some extent)

One-Factor Models

- General Form: $dr = m(r)\,dt + \sigma(r)\,dz$
  - Ito Process:
    - m: drift factor
    - σ: short rate volatility
    - dz: $\varepsilon\sqrt{t}$; $\varepsilon \sim N(0,1)$
- Model characteristics
  - All rates move in same direction, but not by same amount
  - Many different shapes possible (including inverted)
  - Mean reversion can be built in

Desirable Model Feautures

- Consistent with current term structure
- Consistent with yield volatilities
- No negative interest rates
- Allow term structure to move freely
  - Historically, 80% parallel, 11% 'twist'
- Computationally tractable

Model Taxonomy

	Expected change in r m(r)	Mean Reversion	Volatility of r s(r)	Fits Term Str. Yield	Fits Term Str. Vol
Vasicek	a[m - r]	yes	constant	no	no
CIR	a[m - r]	yes	$\sigma\sqrt{r}$	no	no
Brennan & Schwartz	a[b + L - r]	yes	f(r,L)	no	no
Ho & Lee	g(t)	no	constant	yes	no
BDT	f(t,r,σ)	limited	f(time)	yes	yes
Hull & White	a(t)[m(t) - r]	yes	f(time)	yes	yes

Vasicek Model

- Model form

$$dr = \alpha(\bar{r} - r)dt + \sigma dz$$

  - Constant mean
  - α is rate of mean reversion, also constant
  - Short rate volatility is constant

- Bond pricing and yields

$$P(t) = A(t)e^{-rB(t)}$$

$$R(t) = -\frac{\ln[A(t)]}{t} + \frac{B(t)}{t}r \qquad B(t) = \frac{1}{\alpha}(1 - e^{-\alpha t})$$

$$\ln[A(t)] = \frac{R_\infty}{\alpha}(1 - e^{-\alpha t}) - tR_\infty - \frac{\sigma^2}{4\alpha^3}(1 - e^{-\alpha t})^2$$

$$R_\infty = \lim_{t \to \infty} R(t) = \bar{r} - \frac{1}{2}\frac{\sigma^2}{\alpha^2}$$

Vasicek Option Pricing

- Closed form solution (Jamshidian, 1989)

$$c(T,s) = P(s)N(d_1) - XP(T)N(d_2)$$
$$p(T,s) = XP(T)N(-d_2) - P(s)N(-d_1)$$

$$d_1 = \frac{\ln\left(\frac{P(s)}{XP(T)}\right)}{\sigma_p} + \frac{\sigma_p}{2} \quad d_2 = d_1 - \sigma_p \quad \sigma_p = \frac{v(T)(1-e^{-\alpha(s-T)})}{\alpha}$$

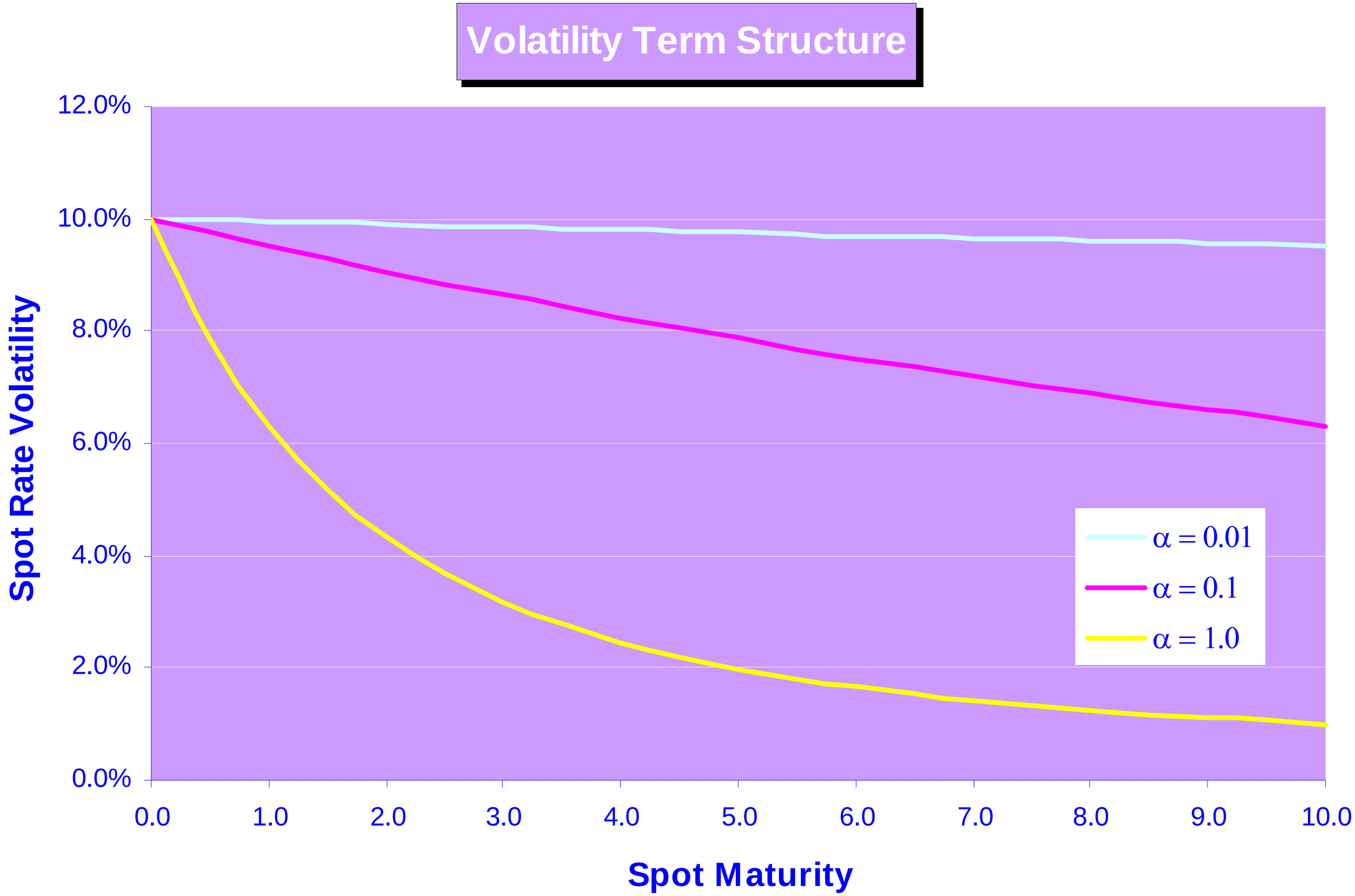
$$v(T) = \sqrt{\frac{\sigma^2(1-e^{-2\alpha T})}{2\alpha}}$$

Vasicek Spot Rate Volatility

- Volatility term structure
  - Determined by σ and α
  - Negative exponential decay
  - Higher mean reversion "dampens" volatility

$$\sigma_R(t) = \frac{\sigma}{\alpha t}(1 - e^{-\alpha t})$$

Volatility Term Structure



Vasicek Example

- 2-year call option on 10-year ZCB
  - r = rbar = 5%
  - Mean reversion parameter α = 0.1
  - σ = 1%, strike X = 0.625

Vasicek Example

$B(T)$	1.8127	$B(s)$	6.321
R_{∞}	0.0450		
$LnA(T)$	-0.0093	$LnA(s)$	-0.17554
$P(T)$	0.9049	$P(s)$	0.6116
$\sigma_R(T)$	0.9063%		
$v(T)$	0.0128	d_1	1.1427
σ_P	7.070%	d_2	1.0720

Option	
$C(T,s)$	**0.0489**

Problems with Vasicek

- Rates can become negative
  - With non-zero probability
- Volatility is constant
  - Empirical evidence suggests that volatility is correlated with the level of interest rates

Cox-Ingersoll-Ross

- Volatility increases with r ½

$$dr = \alpha(\bar{r} - r)dt + \sigma\sqrt{r}dz$$

- Drawbacks
  - Not term structure consistent
  - Not consistent with volatility term structure
  - Curves tend to be monotonically increasing, decreasing or very slightly humped

No Arbitrage Models

- Consistent with term (and volatility) structure
- Examples:
  - Ho and Lee (1986)
  - Black, Derman, Toy (1990)
  - Hull and White (1993)

Ho and Lee

- Originally developed in binomial framework
- Continuous time limit of short rate process:

$$dr = \theta(t)dt + \sigma dz$$

- θ(t)is time-dependent drift, reflecting:
  - Slope of initial forward rate curve
  - Volatility of short rate process

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + \sigma^2 t$$

- Nb: volatility is constant

Ho and Lee Bond Pricing

- Forward discount function

$$P(T,s) = A(T,s)e^{-B(T,s)r(T)}$$

  - B(T,s) = (s-T)

$$LnA(T,s) = \ln\frac{P(0,s)}{P(0,T)} - B(T,s)\frac{\partial \ln P(0,T)}{\partial T} - \frac{1}{2}\sigma^2 TB(T,s)^2$$

Ho & Lee Option Pricing

- $c(T,s) = P(0,s)N(d_1) – XP(0,T)N(d_2)$
- $p(T,s) = XP(0,T)N(-d_2) – P(0,s)N(- d_1)$

$$d_1 = \frac{\ln\left[P(0,s) \big/ XP(0,T) \right]}{\sigma_P} + \frac{\sigma_P}{2} \qquad d_2 = d_1 - \sigma_P$$

$$\sigma_P = \sigma(s-T)\sqrt{T}$$

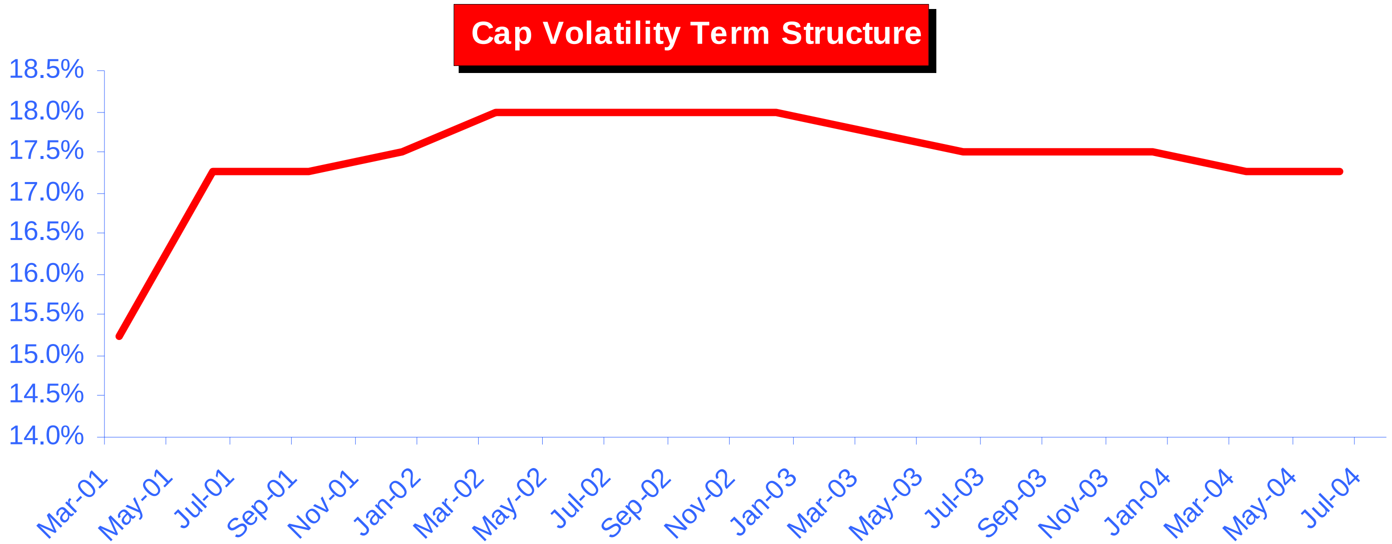
Ho-Lee Model Calibration

- Use market data to calibrate model
  - Find volatility which minimizes (sums of squares) of differences between model option prices and market prices:

$$Min\sqrt{\sum_{i=1}^{N}\left(\frac{P_i-\hat{P}_i}{P_i}\right)^2}$$

Ho-Lee Example

- Given volatilities of 3-month 7% caps
  - Back out prices using Black(76)
  - Calibrate a Ho-lee model



Ho-Lee Solution

- Calibration volatility: 17.6%

Maturity	Cap Term	7% Volatility	Spot Rate	Spot DF	Fwd Rate	Black (76) Caplet Price	Black(76) Cap Price	Ho-Lee Caplet Price	Ho-Lee Cap Price	17.6% Difference
11-Jan-01	0.00			1.0000						
11-Mar-01	0.16	15.25%	6.35%	0.9898	6.66%	0.0001	0.0001	0.0002	0.0002	0.0000
11-Jun-01	0.41	17.25%	6.54%	0.9733	7.31%	0.0012	0.0013	0.0012	0.0014	-0.0001
11-Sep-01	0.67	17.25%	6.83%	0.9555	7.86%	0.0023	0.0037	0.0023	0.0037	-0.0001
11-Dec-01	0.92	17.50%	7.11%	0.9370	8.15%	0.0029	0.0066	0.0029	0.0067	-0.0001
11-Mar-02	1.16	18.00%	7.33%	0.9184	8.12%	0.0030	0.0096	0.0030	0.0097	-0.0001
11-Jun-02	1.41	18.00%	7.47%	0.8998	8.06%	0.0030	0.0126	0.0029	0.0126	0.0000
11-Sep-02	1.67	18.00%	7.56%	0.8817	8.17%	0.0032	0.0157	0.0031	0.0158	0.0000
11-Dec-02	1.92	18.00%	7.64%	0.8639	8.34%	0.0034	0.0192	0.0034	0.0192	0.0000
11-Mar-03	2.16	17.75%	7.72%	0.8463	8.01%	0.0030	0.0221	0.0029	0.0221	0.0000
11-Jun-03	2.41	17.50%	7.75%	0.8294	8.07%	0.0030	0.0252	0.0031	0.0251	0.0000
11-Sep-03	2.67	17.50%	7.78%	0.8127	8.13%	0.0031	0.0283	0.0031	0.0283	0.0000
11-Dec-03	2.92	17.50%	7.81%	0.7964	8.19%	0.0032	0.0315	0.0032	0.0315	0.0000
11-Mar-04	3.16	17.25%	7.84%	0.7803	8.11%	0.0031	0.0346	0.0031	0.0346	0.0000
11-Jun-04	3.42	17.25%	7.86%	0.7645						0.0000

Hull & White Model

- Short Rate Process Model
  - Like Ho-Lee but with mean reversion
  - Like Vasicek fitted to term structure
    - α is mean reversion parameter

$$dr = [\theta(t) - \alpha r]dt + \sigma dz$$

- Drift process

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + \alpha f(0,t) + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

Hull-White Volatility

- HW is same as Vasicek, with time-dependant mean reversion parameter α
- Volatility structure is function of volatility and mean reversion of short rate

$$\sigma_R(t,s) = \frac{\sigma}{\alpha(s-t)}\left(1 - e^{-\alpha(s-t)}\right)$$

Hull-White Discount Function

$$P(T,s) = A(T,s)e^{-B(T,s)r(T)}$$

$$B(T,s) = \frac{1}{\alpha}(1 - e^{-\alpha(s-T)})$$

$$\ln A(T,s) = \ln\frac{P(t,s)}{P(t,T)} - B(T,s)\frac{\partial \ln P(t,T)}{\partial T} - \frac{1}{4\alpha^3}\sigma^2(e^{-\alpha(s-t)} - e^{-\alpha(T-t)})^2(e^{2\alpha(T-t)} - 1)$$

Hull-White Option Pricing

- Modified version of Black-Scholes

$$c(t,T,s) = P(t,s)N(d_1) - KP(t,T)N(d_2)$$

$$p(t,T,s) = KP(t,T)N(-d_2) - P(t,s)N(-d_1)$$

$$d_1 = \frac{\ln(P(t,s)/KP(t,T))}{\sigma_P} + \frac{\sigma_P}{2}$$

$$d_2 = d_1 - \sigma_P$$

$$\sigma_P^2 = \frac{\sigma^2}{2\alpha^3}(1-e^{-2\alpha(T-t)})(1-e^{-\alpha(s-T)})^2$$

Hull-White Calibration

- Calibrate wrt mean reversion and volatility parameters

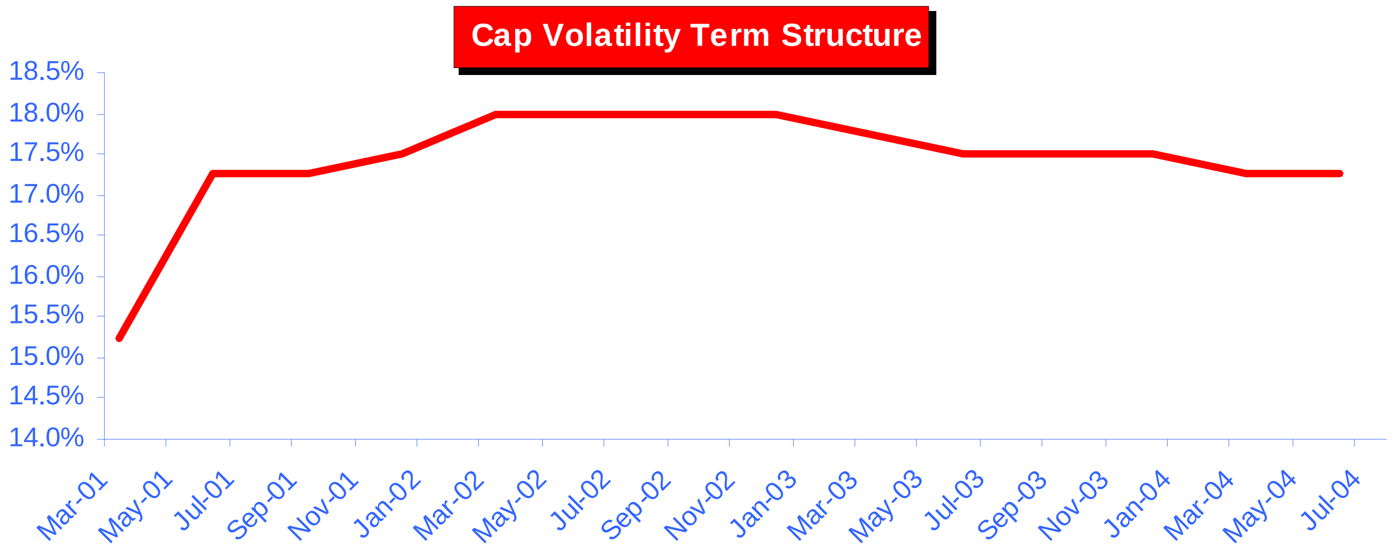
$$\underset{\alpha,\sigma}{Min}\sqrt{\sum_{1}^{N}\left[\frac{C_i^*(\alpha,\sigma)-C_i(\alpha,\sigma)}{C_i(\alpha,\sigma)}\right]}$$

Where,

- C_i is the market price of option I
- C^*_i is the model price of option I

Hull-White Example

- Given volatilities of 3-month 7% caps
  - Back out prices using Black(76)
  - Calibrate a Ho-lee model



Hull-White Solution

α	-1.16%
σ	69.53%

Maturity	Cap Term	7% Volatility	Spot Rate	Spot DF	Fwd Rate	Black (76) Caplet Price	Black(76) Cap Price	Hull White Caplet Price	Hull White Cap Price	Difference
11-Jan-01	0.00			1.0000						
11-Mar-01	0.16	15.25%	6.35%	0.9898	6.66%	0.0001	0.0001	0.0002	0.0002	0.0000
11-Jun-01	0.41	17.25%	6.54%	0.9733	7.31%	0.0012	0.0013	0.0012	0.0014	-0.0001
11-Sep-01	0.67	17.25%	6.83%	0.9555	7.86%	0.0023	0.0037	0.0023	0.0037	-0.0001
11-Dec-01	0.92	17.50%	7.11%	0.9370	8.15%	0.0029	0.0066	0.0029	0.0067	-0.0001
11-Mar-02	1.16	18.00%	7.33%	0.9184	8.12%	0.0030	0.0096	0.0030	0.0097	0.0000
11-Jun-02	1.41	18.00%	7.47%	0.8998	8.06%	0.0030	0.0126	0.0029	0.0126	0.0000
11-Sep-02	1.67	18.00%	7.56%	0.8817	8.17%	0.0032	0.0157	0.0031	0.0157	0.0000
11-Dec-02	1.92	18.00%	7.64%	0.8639	8.34%	0.0034	0.0192	0.0034	0.0191	0.0001
11-Mar-03	2.16	17.75%	7.72%	0.8463	8.01%	0.0030	0.0221	0.0030	0.0221	0.0001
11-Jun-03	2.41	17.50%	7.75%	0.8294	8.07%	0.0030	0.0252	0.0031	0.0251	0.0000
11-Sep-03	2.67	17.50%	7.78%	0.8127	8.13%	0.0031	0.0283	0.0031	0.0283	0.0000
11-Dec-03	2.92	17.50%	7.81%	0.7964	8.19%	0.0032	0.0315	0.0032	0.0315	0.0000
11-Mar-04	3.16	17.25%	7.84%	0.7803	8.11%	0.0031	0.0346	0.0032	0.0347	-0.0001
11-Jun-04	3.42	17.25%	7.86%	0.7645						**0.0000**

Black-Derman-Toy Model

- Simple one-factor model
  - All rates stochastic
  - Volatilities can change over time
- No negative interest rates
- Fits the yield curve and yield volatility term structure
- Easy to use & implement

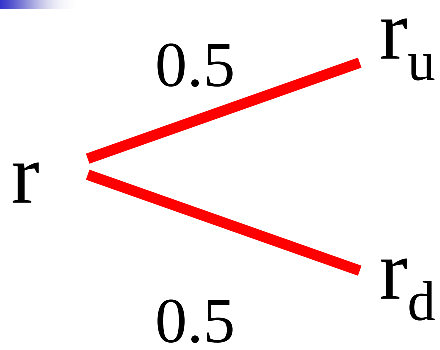
Black-Derman-Toy Model

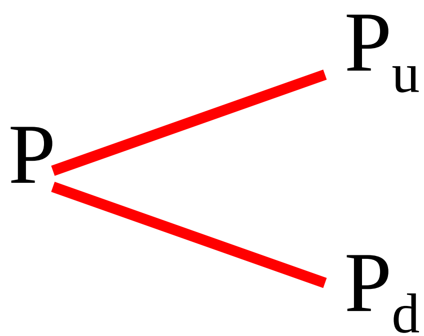
- Inputs:
  - Current spot yield curve (ZCB prices)
  - Spot rate volatilities
- Output:
  A binomial short rate tree that matches:
  - Term structure of zero coupon yields
  - Term structure of spot rate volatility
- This can then be used to price:
    - Bonds
    - Derivatives
    - Any interest rate contingent claim

Model Form & Assumptions

- Model : $$d\ln(r) = \left[\theta(t) + \frac{\sigma'(t)}{\sigma(t)}\ln(r)\right]dt + \sigma(t)dz$$
- Short rates are log-normally distributed:
  - Yields are always positive
  - Volatility of log of short rate depends only on time
  - Probability up and down moves is 50%
- Expected return on all assets is equal
  - A single short rate holds in each future period

The Short Rate Tree

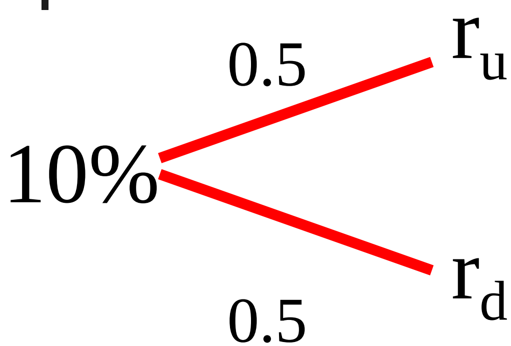


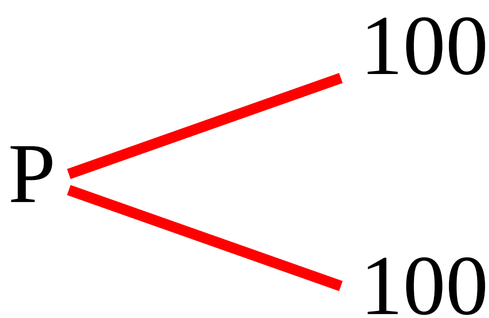


$$P = \frac{[0.5\,P_u + 0.5\,P_d]}{(1 + r)}$$

Note: Need P_u and P_d but not r_u or r_d

The Short-Rate Tree Example 1-Perdiod ZCB

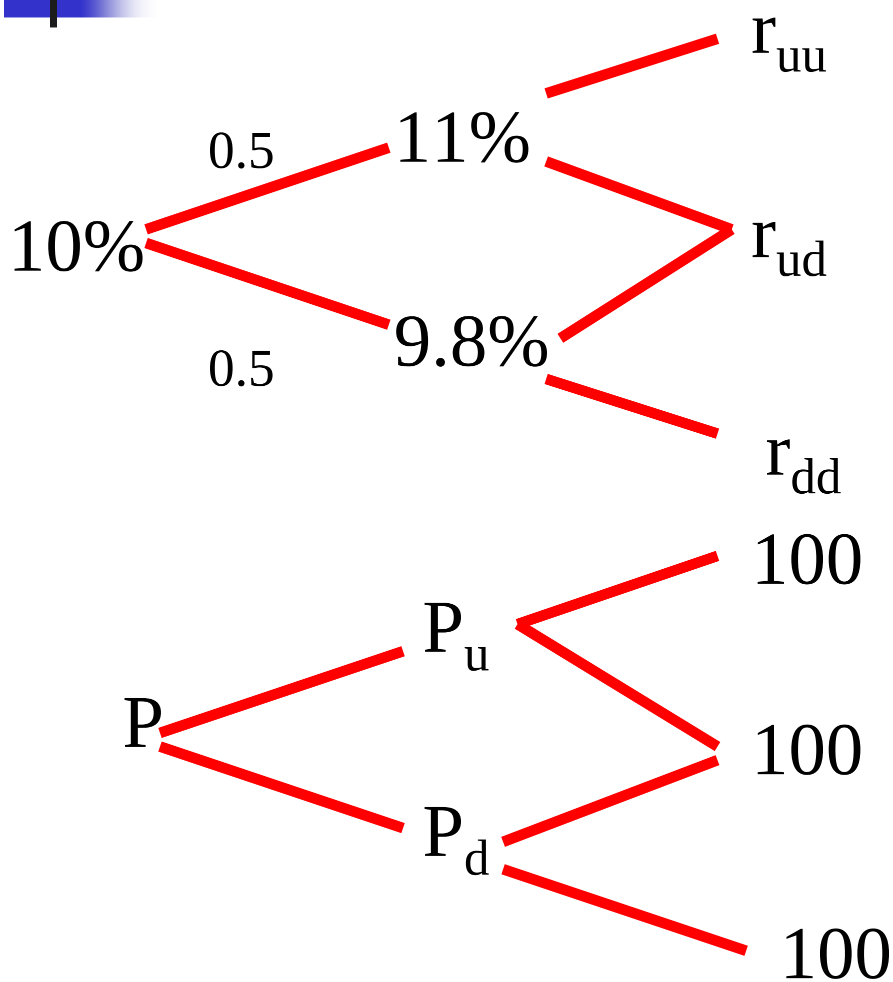




$$P = \frac{[0.5\;100 + 0.5\;100]}{(1 + 0.1)}$$

$$P = 90.91$$

Example: 2 Year ZCB



$$P_u = \frac{[0.5\ 100 + 0.5\ 100]}{(1 + .11)}$$

$$P_u = 90.09$$

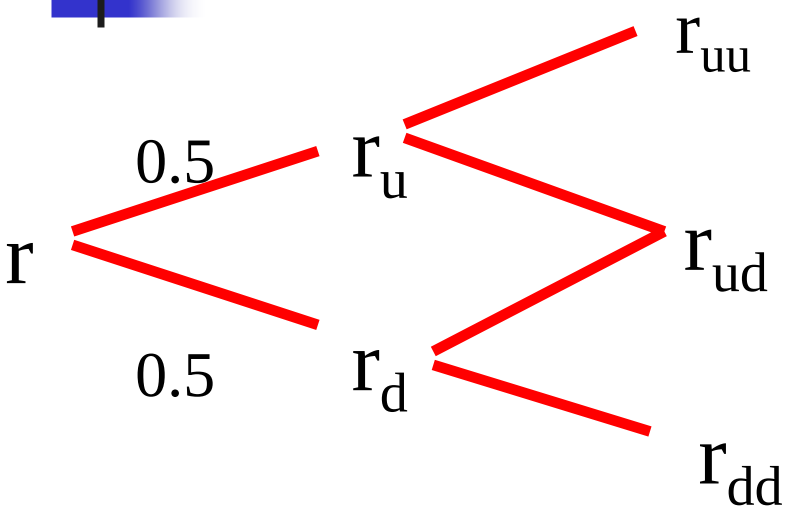
$$P = \frac{[0.5\ 90.09 + 0.5\ 91.07]}{(1 + .1)}$$

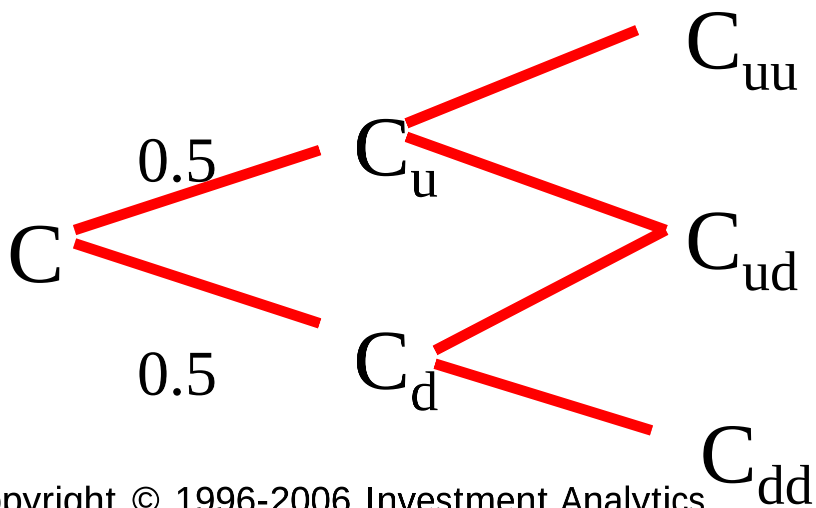
$$P = 82.35$$

$$P_d = \frac{[0.5\ 100 + 0.5\ 100]}{(1 + .098)}$$

$$P_d = 91.07$$

Vanilla Swap





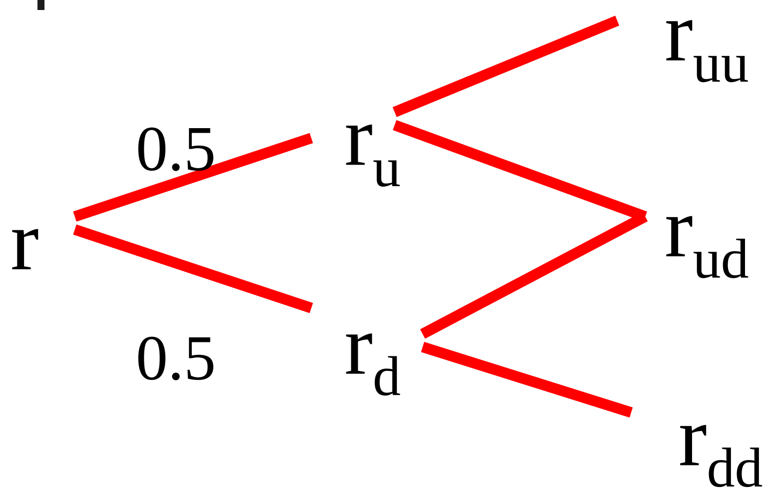
Swap payments in arrears:

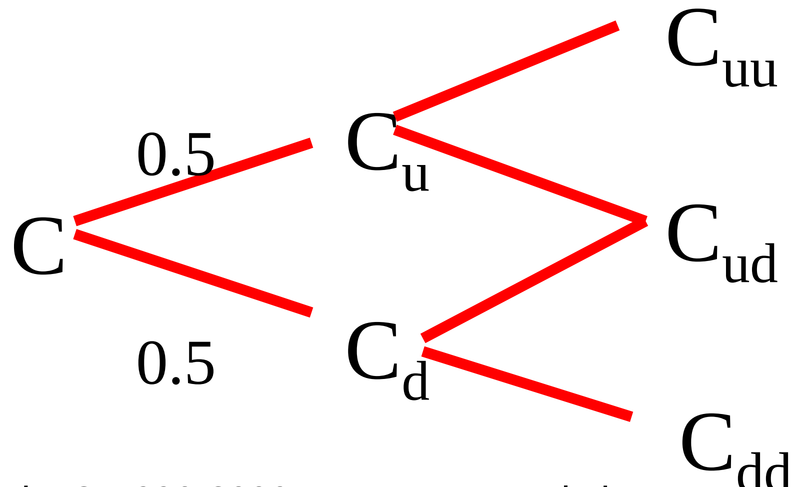
$C_{uu} = NP[r_u\text{-}c]$,
$C_{ud} = NP[r_u\text{-}c]$, $C_{dd} = NP[r_d\text{-}c]$
$C_u = NP[r\text{-}c]$, $C_d = NP[r\text{-}c]$,
$C = 0$

To value, work back, adding cash flows at current node:

$$V_u = C_u + \frac{[0.5V_{uu} + V_{ud}]}{(1 + r_u)}$$

Cap





Cap payments in arrears:

$C_{uu} = C_{ud} = NP \times Max[r_u - r_x, 0]$,
$C_{dd} = NP \times Max[r_d - r_x, 0]$
$C_u = C_d = 0$

To value, work back, adding cash flows at current node:

$$V_u = C_u + \frac{[0.5V_{uu} + V_{ud}]}{(1 + r_u)}$$

Valuing Other Securities

- Same principles apply for any fixed income securities
  - Bonds, bond options, swaptions
    - Embedded options, e.g. callable bonds, CMO's
  - Caps, floors, collars
  - FRA's, swaps, forwards, FRN's
  - Futures
    - Can even calculate convexity bias since know cash flow at every node

Constructing the Tree

- Choose short rate tree so that:
  - Calculated ZCB prices (spot rates) agree with market prices (spot rates)
  - ZCB yield volatility agrees with input volatility
- Procedure
  - Guess a new column of short rates
  - Calculate the price of the next period out ZCB
  - Compare:
    - calculated ZCB price with market price
    - yield volatility from price tree with input volatility
  - Modify guess, and repeat until matches

Filling a New Column in the Short Rate Tree

- Guess two numbers:
  - r_{min}: the short rate at the bottom node
  - σ_r : the short rate volatility in the column (Note: this is NOT the yield volatility we are trying to match)
- Short Rate Volatility:
  - $\sigma_r = 0.5 * \text{Ln}\ (r_u / r_d)$
- Use $r_u = r_d\exp(2\sigma_r)$ to step up the column node by node

Example

Col 4 Col 5

- 21.79
- 16.06
- 11.83
- 8.72

-
-
-
- 12.15
- 8.65

r_{min}

Next node up:

$r_u = 0.865 \times \exp(2 \times 0.17)$
$= 0.1215$

Guess $r_{min} = 0.0865$
$\sigma_r = 0.17$

Example: Using 3-Year ZCB to Calibrate Short Rate Tree

- We know:
  - Short rate tree for first two periods
  - Market price of the 3-year ZCB (71.1768)
  - Volatility of the 3-year ZCB yield (18%)

Example: Using 3-Year ZCB to Calibrate Short Rate Tree

- Before:

Price				
				100.00
			100.00	
Market	**Model**	87.47		100.00
71.18	**81.16**		100.00	
		91.08		100.00
			100.00	
				100.00

Rmin	
σ	

Yield Vol	
Market	**Model**
18.00%	**18.50%**

		0.00%
	14.32%	
10.00%		0.00%
	9.79%	
		0.00%

		0.00%
	6.92%	
7.21%		0.00%
	4.78%	
		0.00%

Example: Using 3-Year ZCB to Calibrate Short Rate Tree

- Guess: r_{min} = 9.76%, σ = 17.19%

Price				
				100.00
			83.74	
Market	Model	75.07		100.00
71.18	71.18		87.90	
		81.52		100.00
			91.11	
				100.00

Rmin	9.76%
σ	17.19%

Yield Vol	
Market	Model
18.00%	18.00%

		19.41%
	14.32%	
10.00%		13.76%
	9.79%	
		9.76%

		19.41%
	15.41%	
12.00%		13.76%
	10.75%	
		9.76%

Checking the Yield Volatility

- Compute the ZCB yields (spot rates)

$$y_{u2} = [100/75.067]^{1/2} - 1 = 15.418\%$$

$$y_3 = [100/71.1768]^{1/3} - 1 = 12\%$$

$$y_{d2} = [100/81.521]^{1/2} - 1 = 10.755\%$$

- Compute yield volatility & compare with market data:
  - $0.5 \ln (y_u / y_d) = 0.5 \ln(0.15418/.10755) = 18\%$

Lab: Black-Derman-Toy

- Worksheet: BDT Option Pricing
- Required:
  - Construct short rate tree 1994-2005 (semiannual)
  - Price a ZCB maturing May 2005
  - Price 12% of '05
  - Price a call option on the 12% of '05, strike 100, maturity May 1999
  - Compute option delta
- See Notes & Solution
- Then: Exercises for the BDT model
  - See instructions in folder

Limitations of One-Factor Models

- These are one-factor models
  - Yield curve evolution quite limited
  - However, simple to estimate & use
- Have to re-calibrate frequently
  - Parameter stability questionable
- Markovian Property
  - Evolution of short rate does nor depend on previous behavior

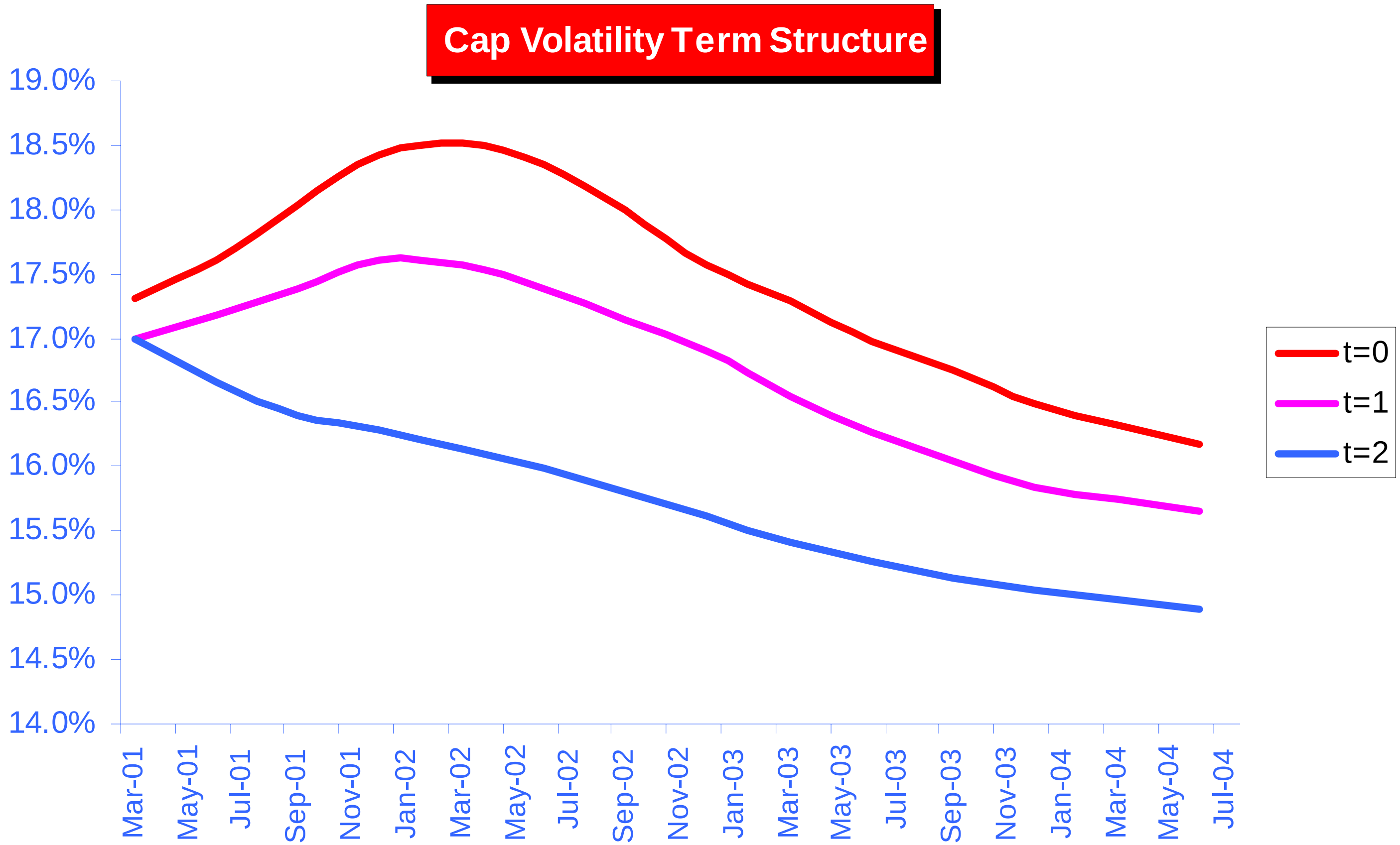
Volatility Process in One-Factor Models

- Volatility process is constrained
  - Spot rate volatility at time T depends on:
    - S-maturity and T-maturity yield volatilities
    - Slope of volatility curve at maturity T (t <= T <= s)
  - Result is that volatility curve tends to lose definition over time

$$\sigma_R(T,s) = \frac{\sigma}{(s-t)} \frac{\left[(s-t)\sigma_R(t,s) - (T-t)\sigma_R(t,T)\right]}{\left[(T-t)\frac{\partial \sigma_R(t,T)}{\partial T} + \sigma_R(t,T)\right]}$$

$$\sigma_R(T,s) = \frac{\sigma}{(s-t)} \frac{\left[(s-t)\sigma_R(t,s) - (T-t)\sigma_R(t,T)\right]}{\left[(T-t)\frac{\partial \sigma_R(t,T)}{\partial T} + \sigma_R(t,T)\right]}$$

Volatility Process Example



Two Factor Models

- Limitations of one-factor models
  - Yield curve tend to be monotonic, or slightly humped
- Two-Factor models are "richer"
  - Especially volatility process
- Examples
  - Longstaff & Schwartz
  - Hull & White
  - Heath Jarrow Morton

Fong & Vasicek

- Two factors
  - Short rate, r
  - Short rate variance v

$$dr = [\alpha(\bar{r} - r)]dt + \sqrt{v}dz_1$$

$$dv = [\gamma(\bar{v} - v)]dt + \xi\sqrt{v}dz_2$$

$$with \quad dz_1 dz_2 = \rho dt$$

- Closed formula for bond, option prices
  - See also Selby & Strickland (1992)

Longstaff & Schwartz

- Characteristics
  - One of few models that provides closed form solutions when volatility is stochastic
- Equilibrium Model
  - $r = ax + by; \quad a <> b$
  - $v = a^2x + b^2y$
- State variables x & y follow stochastic DE's:

$$dx = (\gamma - \delta x)dt + \sqrt{x}dz_1$$

$$dy = (\eta - \theta y)dt + \sqrt{y}dz_2$$

Longstaff & Schwartz

- Explicit formula for:
  - Discount bond prices
  - Derivatives
  - Volatility term structure
- Volatility term structure
  - Two factors
  - Allows greater variety of structures than one factor models

Longstaff & Schwartz Calibration

- Very rich model
- Lots of possible calibration criteria:
  - Quality of fit to market yield curves
  - Stability of coefficients
  - Match observed correlation between rates

Longstaff & Schwartz – Pros & Cons

- Advantages
  - Many different shapes of term structure
  - Complex volatility term structures possible
  - Correlation between rates can vary
    - Short rate not perfectly correlated with its volatility
    - If it were, all rates would be perfectly correlated (as BDT)
  - Close form solutions
- Disadvantages
  - Complexity
  - Parameterization
  - Calibration

Hull & White 2-Factor Model

- Model form

$$dr = [\theta(t) + u - \alpha r]dt + \sigma_1 dz_1$$

$$du = -budt + \sigma_2 dz_2$$

  - u is random mean reversion level
- Characteristics
  - Similar to original HW model
    - Time dependent function $\theta(t)$
  - Extra flexibility of drift term due to u
    - Richer term structure evolutions and volatility structures

H-W Volatility Term Structure

$$\sigma_R(t,s) = \frac{1}{(s-t)}\sqrt{\left[(B(t,s)\sigma_1)^2 + (C(t,s)\sigma_2)^2 + 2\rho\sigma_1\sigma_2 B(t,s)C(t,s)\right]}$$

$$B(t,s) = \frac{1}{\alpha}(1-e^{-\alpha(s-t)})$$

$$C(t,s) = \frac{1}{\alpha(\alpha-b)}e^{-\alpha(s-t)} - \frac{1}{b(\alpha-b)}e^{-b(s-t)} + \frac{1}{\alpha b}$$

Hull-White Option Pricing

- Same modified Black-Scholes model
  - More complex formula for σ_P

$$\sigma_P^2 = \frac{\sigma_1^2}{2\alpha} B(T,s)^2 (1 - e^{-2\alpha(T-t)}) +$$

$$\sigma_2^2 \left[\frac{U^2}{2\alpha} e^{-2\alpha(T-t)} + \frac{V^2}{2b}(e^{2b(T-t)} - 1) - 2\frac{UV}{\alpha+b}(e^{(\alpha+b)(T-t)} - 1) \right] +$$

$$\frac{2\rho\sigma_1\sigma_2}{\alpha}(e^{-\alpha(T-t)} - e^{-\alpha(s-t)}) \left[\frac{U}{2\alpha}(e^{2\alpha(T-t)} - 1) - \frac{V}{\alpha+b}(e^{(\alpha+b)(T-t)} - 1) \right]$$

$$U = \frac{1}{\alpha(\alpha-b)} \left[e^{-\alpha(s-t)} - e^{-\alpha(T-t)} \right] \qquad V = \frac{1}{b(b-\alpha)} \left[e^{-b(s-t)} - e^{-b(T-t)} \right]$$

Hull-White Pros and Cons

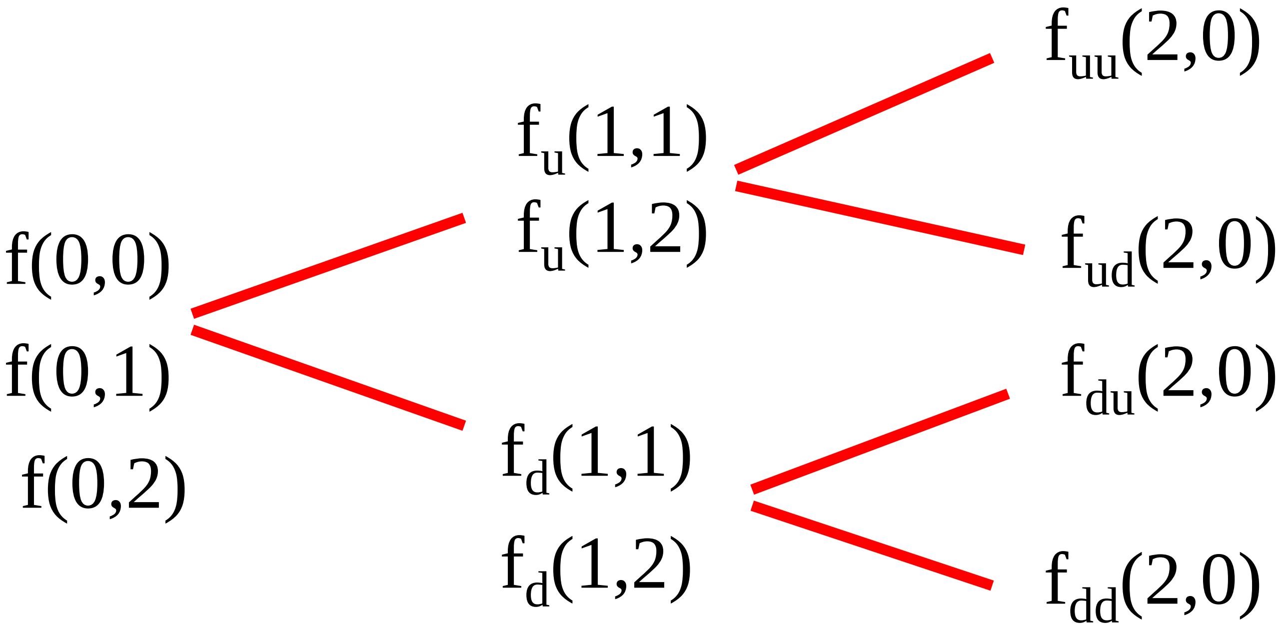
- Advantages
  - More complex term and volatility structures
  - Closed form solutions
  - Important characteristics like mean reversion and yield correlations can be modeled
  - Calibrates well with caps & floors
- Disadvantages
  - Rates can in theory become negative

Heath-Jarrow-Morton Model

- HJM framework: very general class of models
- Model *forward* rates, not short rate
  - Entire forward curve moves in the tree
  - Much more general than just short-rate moving

The Forward Rate Tree

- Let f(t,T) = forward rate at time t between T and T+1



HJM Arbitrage

- Question:
  - What are allowable movements in forward curve?
  - HJM show that movements cannot be arbitrary
    - Otherwise tree is not arbitrage free

HJM Arbitrage Restrictions

- HJM worked restrictions on evolution of forward curve to prevent arbitrage
- Basically, f moves to f_u or f_d
  - $(f_u - f)$ and $(f_d - f)$ depend only on volatility of forward rates
- Once volatilities and initial forward curve are known, entire tree is determined.

General Form of HJM

- Stochastic Differential Equation

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{n} \sigma_i(t,T, f(t,T))dz_i(t)$$

  - Volatility functions σ_i can depend on entire forward rate curve
  - Drift term

$$\alpha(t,T) = \sum_{i=1}^{n} \left\{ \sigma_i(t,T, f(t,T)) \left[\int_t^T \sigma_i(t,u, f(t,T))du \right] \right\}$$

HJM in Terms of Bond Returns

- HJM can be restated in terms of pure discount bond price returns

$$\frac{\partial P(t,T)}{P(t,T)} = r(t)dt + \sum_{i=1}^{n} v_i(t,T)dz_i(t)$$

- Volatilities of forward rates and bond price:

$$v_i(t,T) = -\int_t^T \sigma_i(t,u)du$$

HJM Model Characteristics

- Calibrated to initial forward rate curve
  - Rates evolve randomly thorough time
  - Volatilities and correlations as specified by the volatility functions
- Generality
  - Handles very general volatility term structures
- Methodology
  - Multinomial trees
  - Monte-Carlo simulation

HJM Model Types

- Volatility Specification
  - $\sigma(t,T) = \sigma$
    - Gives the Ho-Lee model
  - $\sigma(t,T) = \sigma e^{-k(T-t)}$
    - Hull-White model
  - $\sigma(t,T) = \sigma f(t,T)$
    - Forward model, forward rates can explode
    - Bound: $\sigma(t,T) = \sigma \min\{f(t,T),M\}$
  - Market Model
    - Specify volatility etc in terms of simple interest rates
    - Useful for LIBOR based contracts

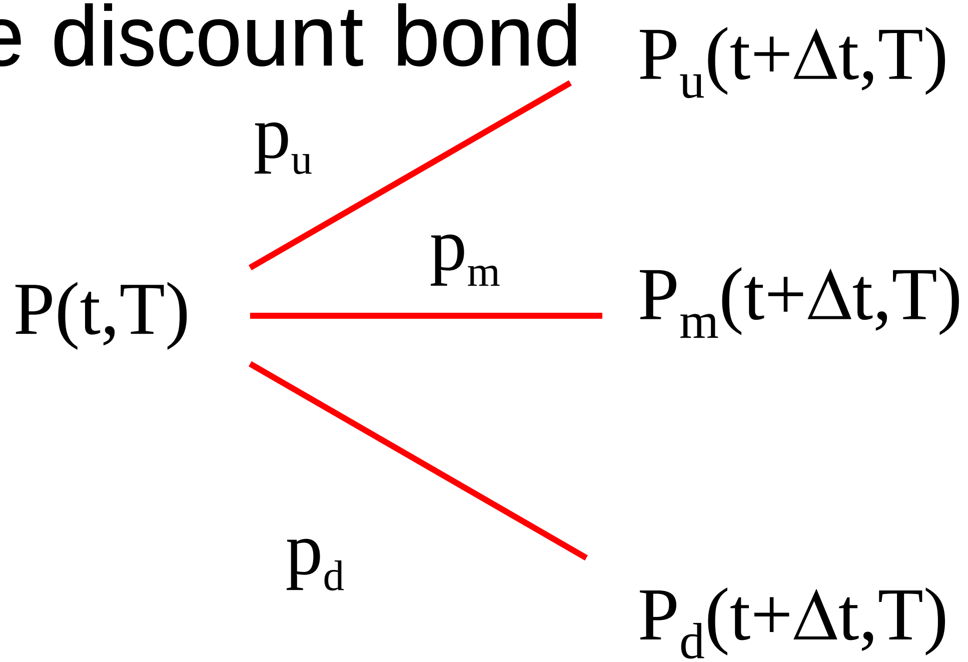
HJM Implementation

- Pricing a bond option with HJM
  - T-maturity option on
  - S-maturity bond
- Methodology
  - Choose discrete points on forward curve
    - Calibrated from market data
  - Evolve these through time until option maturity
  - Find maturity value of option from bond price at time T:

$$P(T,s) = \exp(-\int_T^s f(T,u)du$$

Trinomial Trees for 2-factor Gaussian HJM Model

- Pure discount bond



Gaussian HJM 2-Factor Trinomial Tree

- State equations

$$P_u(t+\Delta t, s) = P(t,s)\exp\left\{\alpha_P(t,s)\Delta t + v_1(t,s)\sqrt{\Delta t}\right\}$$

$$P_m(t+\Delta t, s) = P(t,s)\exp\left\{\alpha_P(t,s)\Delta t - v_1(t,s)\sqrt{\Delta t} + \sqrt{2}v_2(t,s)\sqrt{\Delta t}\right\}$$

$$P_d(t+\Delta t, s) = P(t,s)\exp\left\{\alpha_P(t,s)\Delta t - v_1(t,s)\sqrt{\Delta t} - \sqrt{2}v_2(t,s)\sqrt{\Delta t}\right\}$$

- Pricing
  - Generate pure discount bond prices at each node
  - Hence option payoff at each node
- Non-Recombining
  - Hence 3^N nodes after n steps!

HJM and Monte-Carlo Methods

- Tree methodology very computationally intensive
- Hence Monte-Carlo typically preferred
- Basic idea:
  - Generate large number of term structure paths
  - Also volatility term structure paths
  - Evaluate bond or derivative for each iteration

HJM Option Formula

- European call option

$$c(t,T,s) = E_t\left[\exp\left\{-\int_t^T r_t d\tau\right\}\max(0, P(T,s) - K)\right]$$

- Implement this via Monte-Carlo simulation

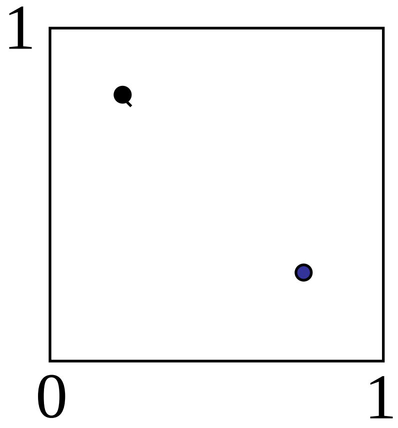
$$c(t,T,s) = \frac{1}{M}\sum_{j=1}^{M}\left[\max(0, P(t,s)Y_j(t,T,s) - KP(t,T)Y_j(t,T,T))\right]$$

$$Y_j(t,T,\tau) = \exp\left[\sum_{i=1}^{N} v(u_i\tau)\varepsilon(u_i)\sqrt{\Delta t} - \frac{1}{2}v(u_i,\tau)^2\Delta t\right]$$

Variance Reduction Techniques: Antithetic Variates

- These are used to **speed convergence** of the Monte Carlo approximation and the most popular are the following
- **Antithetic Variates** Use both u and 1-u to double sample size cheaply

d:=2



- As log as covariance between V{z} and V{1-z} is negative, the overall variance will be substantially reduced

$$V_{est} = \frac{1}{2}\left[V_{est}\{u\} + V_{est}\{1-u\}\right]$$

Variance Reduction Techniques: Control Variates

- **Control Variates** Correct Monte Carlo estimate of exotic value with vanilla MC error

$$\hat{V}^{E} = V^{EMC} + (V^{BSMC} - V^{BS})$$

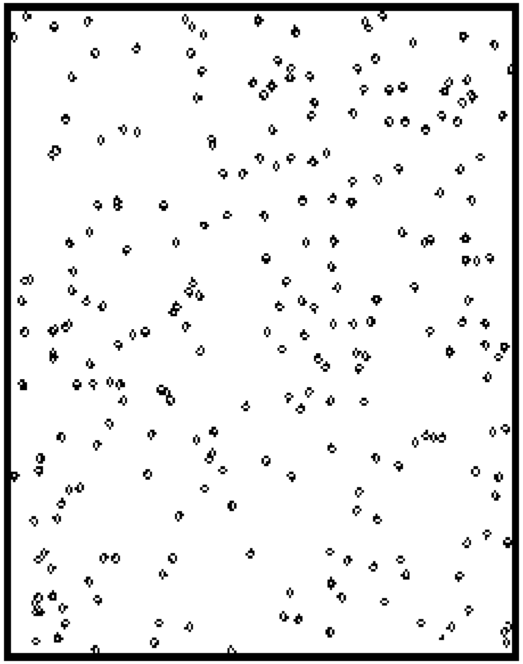
- Variate – Control Variate correlation
  - Reduces estimate variance when control and variate are correlated
    - Based on cancellation of shared estimation errors
  - No benefit if control is uncorrelated with variate,
  - If negative correlated, may increase estimate variance!

Quasi-Random Numbers

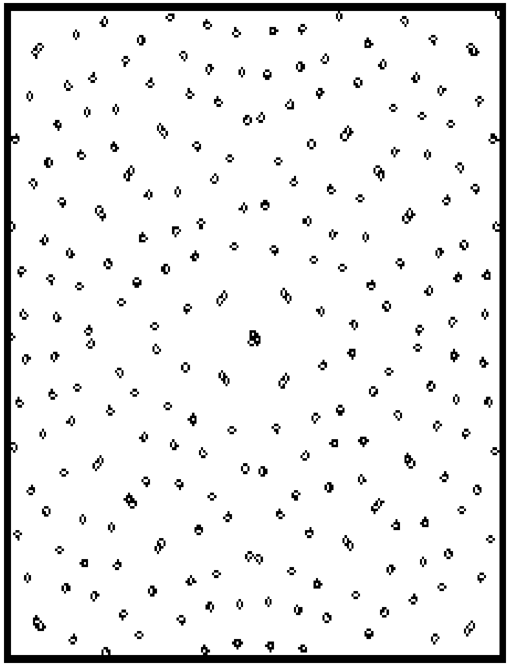
- Low Discrepancy Sequences
- Deterministic sequences generated by number theory
  - **Halton , Sobel, Faure**
  - Sequences appear random, but not “clumpy”
  - Behavior is ideal for fast convergence

Random vs. Sobel

Paskov (1997)

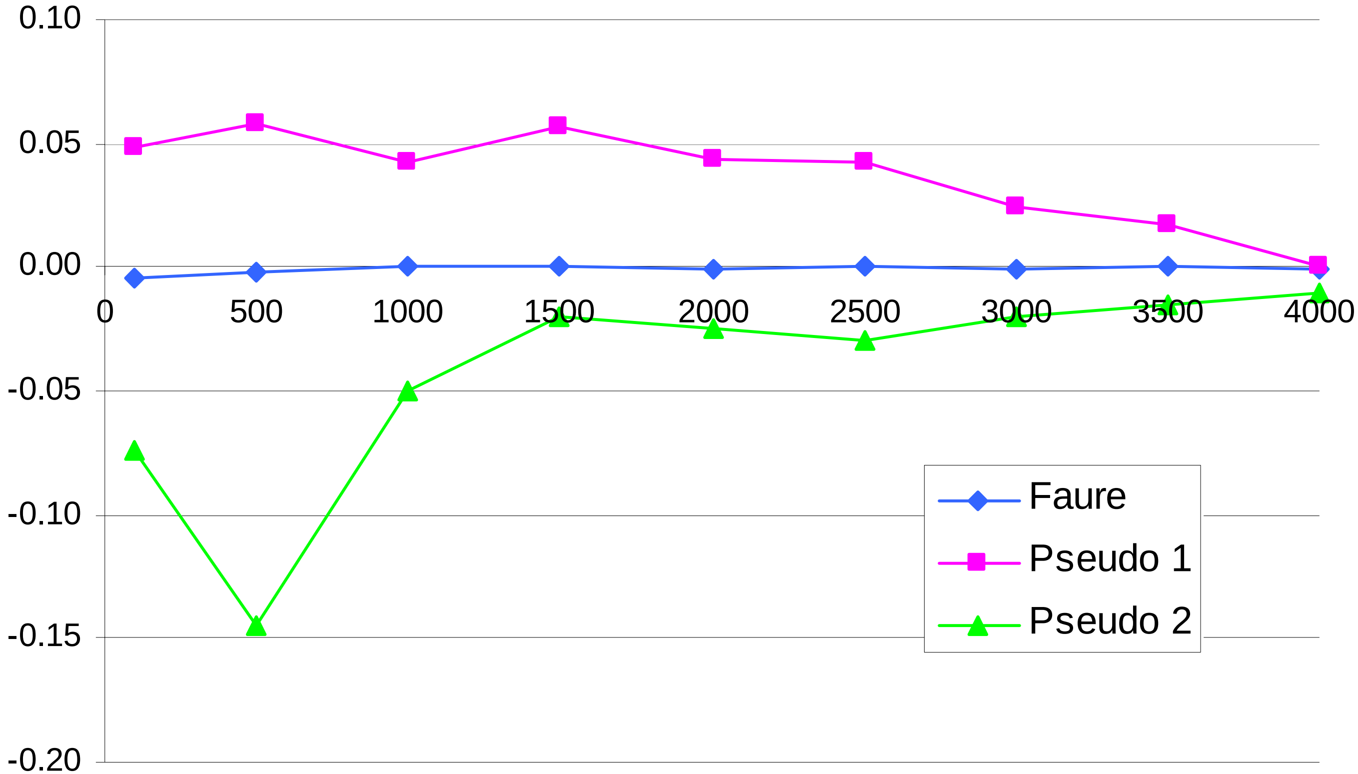


Random points
in the
unit square



Sobol points
in the
unit square

Pricing Error for a European Call



Sensitivity Factors with MCS

- Approximate using finite difference ratios
- Delta $$\frac{\partial C}{\partial P} \approx \frac{C(P+\Delta P) - C(P-\Delta P)}{2\Delta P}$$
- Gamma $$\frac{\partial^2 C}{\partial P^2} \approx \frac{C(P+\Delta P) - 2C(P) + C(P-\Delta P)}{\Delta P^2}$$
- Vega $$\frac{\partial C}{\partial \sigma} \approx \frac{C(\sigma+\Delta \sigma) - C(\sigma-\Delta \sigma)}{2\Delta \sigma}$$
- Theta $$\frac{\partial C}{\partial t} \approx \frac{C(t+\Delta t) - C(t-\Delta t)}{2\Delta t}$$

HJM Models Pros and Cons

- Models typically have recombining trees
  - Estimation, fitting can be hard
- Two factor and multi-factor models can be developed
- Powerful methodology - industry standard

Summary: Interest Rate Models

- One-Factor Models
  - Strengths
    - Simple, often easy to calibrate
  - Weaknesses
    - Range of term & volatility structures limited
- Two-Factor Models
  - Strengths
    - Flexible, powerful & wide range of behaviors
  - Weaknesses
    - Complex, computationally demanding