Forecasting Financial Markets

ARCH Models

Overview

- Asset Return Behavior
- Models
  - ✓ ARCH
  - ✓ GARCH
  - ✓ GARCH-M
- Estimation procedures
- Applications

Defining Volatility

- Nonparametric Definition
  - ✓X is more volatile than y if:
  - ✓$P(|X|>c) > P(|Y|>c)$ for all c
- Time Series Volatility
  - ✓Time series y_t become more volatile when
    - $P(| y_{t+1} | >c) > P(| y_t | > c)$ for all c
    - Occurs if and only if $\sigma_{t=1} > \sigma_t$

Estimating Historical Volatility

- Standard Deviation

$$X_i = Ln(P_{i+1}/P_i)$$

$$\sigma = \sqrt{\sum (X_i - \overline{X})^2/(N-1)}$$

- Parkinson (5x times more efficient)

$$\sigma = \frac{1}{2N\sqrt{Ln(2)}} \sum Ln(H_i / L_i)$$

- Garman & Klass (7 x efficiency)

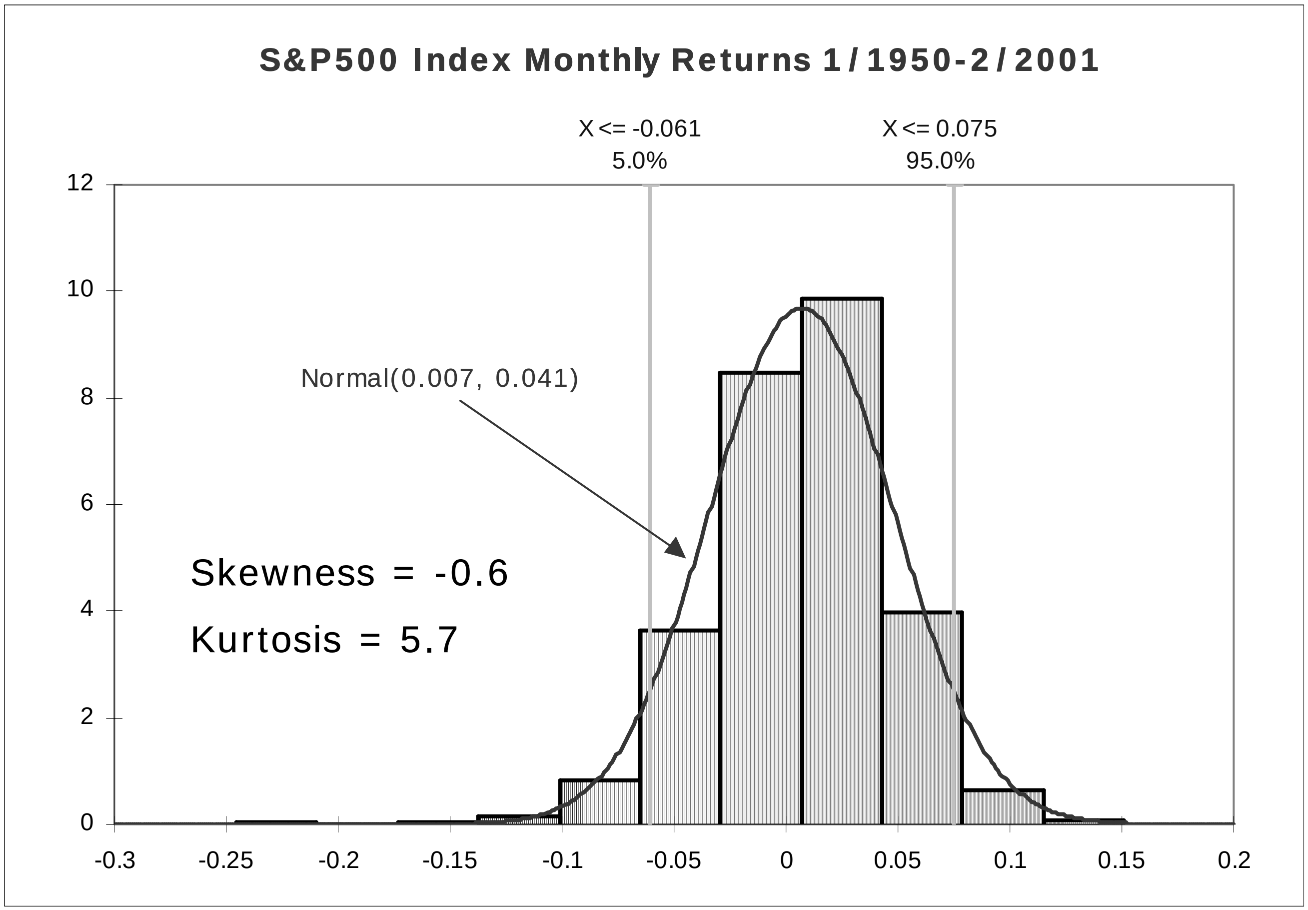
$$\sigma = \sqrt{ABS[\frac{1}{N}\sum \frac{1}{2}[Ln(H_i / L_i)]^2 - \frac{1}{N}\sum (2Ln(2)-1)[Ln(C_i / C_{i-1})]^2]}$$

Asset Return Characteristics

- Thick Tails
- Non-Normal Distribution
- Volatility Clustering
- Leverage
- Trading vs. Non Trading Periods
- Forecastable Events
- Volatility & Correlation

Thick Tails, Non-Normal Distribution

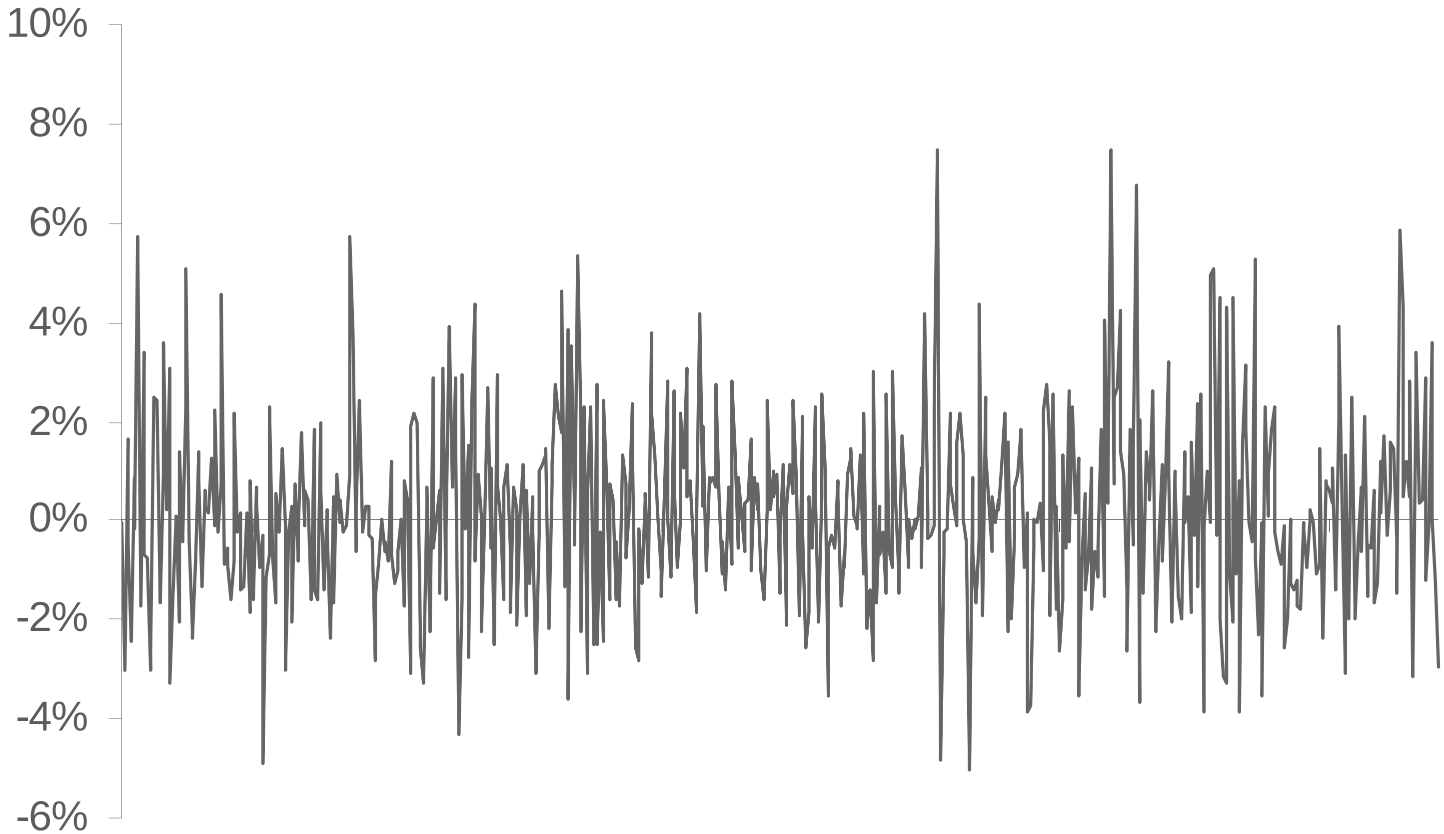
✓Mandelbrot (1963), Fama (1963, 1965)



Volatility is Stochastic

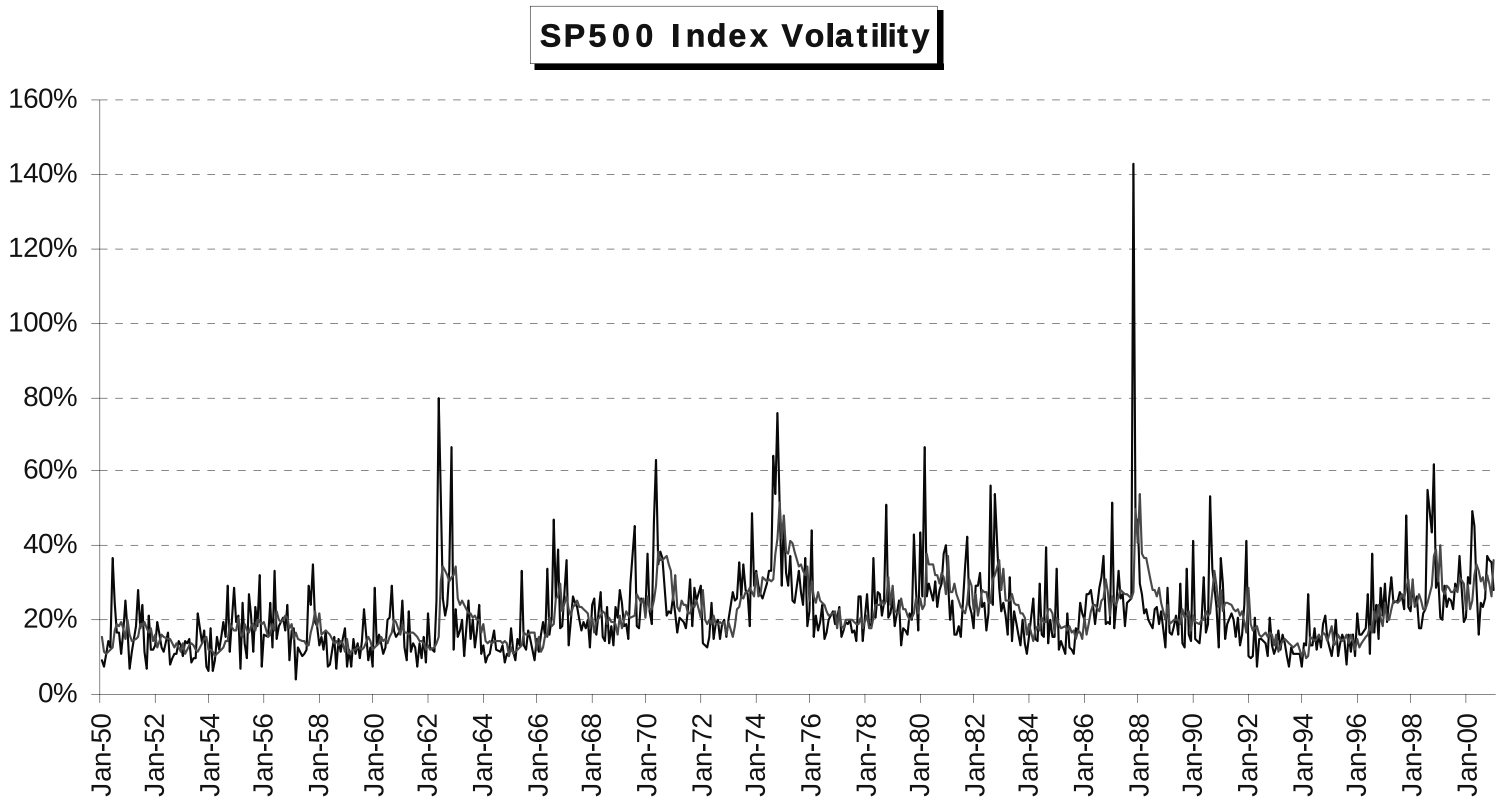
- Non-Constant Variance
  - ✓ Changing conditional variance
  - ✓ Highly volatile periods
  - ✓ Interspersed with quiet periods
  - ✓ Large returns tend to be followed by large returns (positive or negative)

A Conditionally Heteroscedastic Series

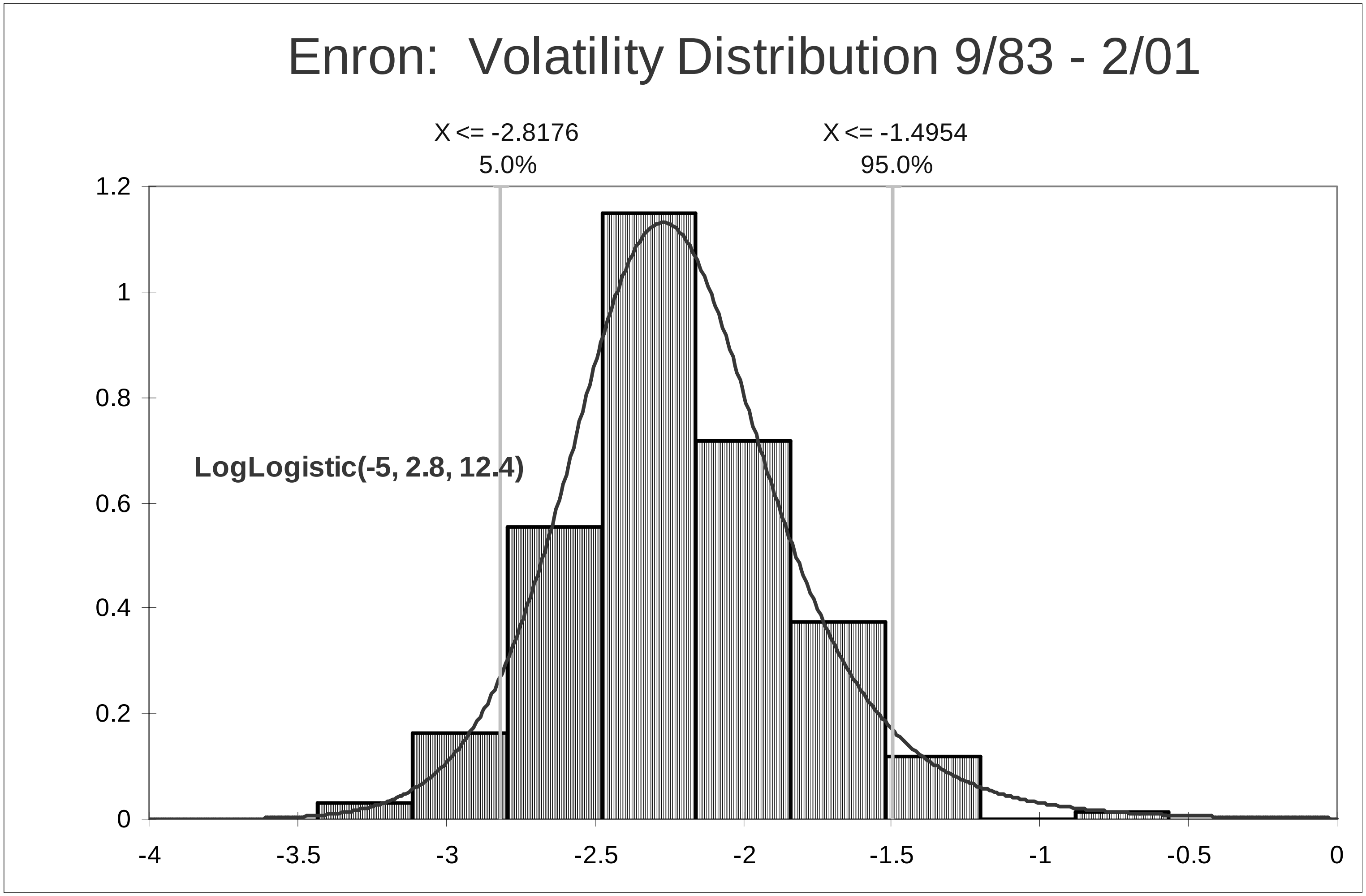


Volatility Clustering

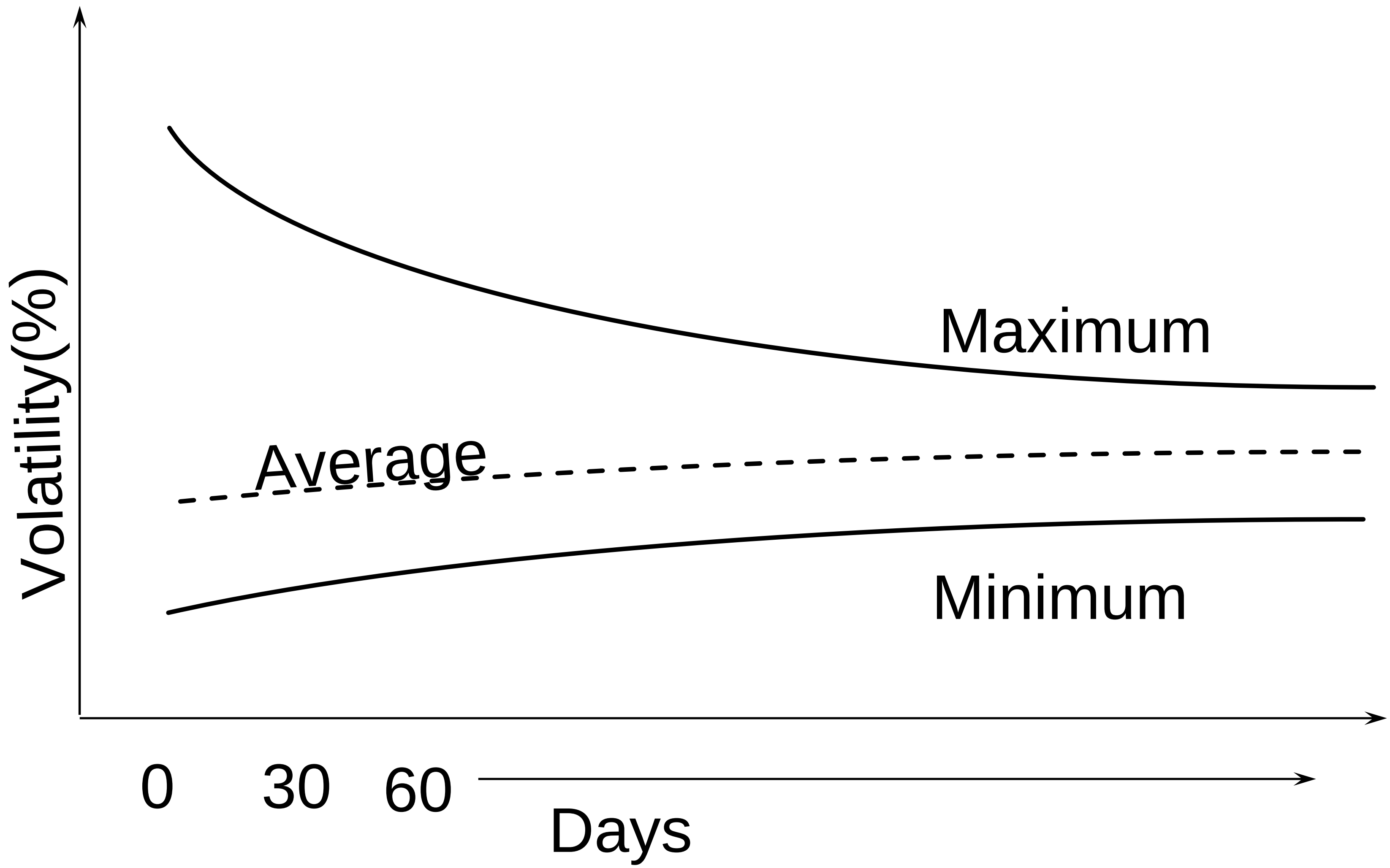
- ✓Periods of high vol. followed by high vol.
- ✓Followed by decay back to normal levels



Volatility Distributions



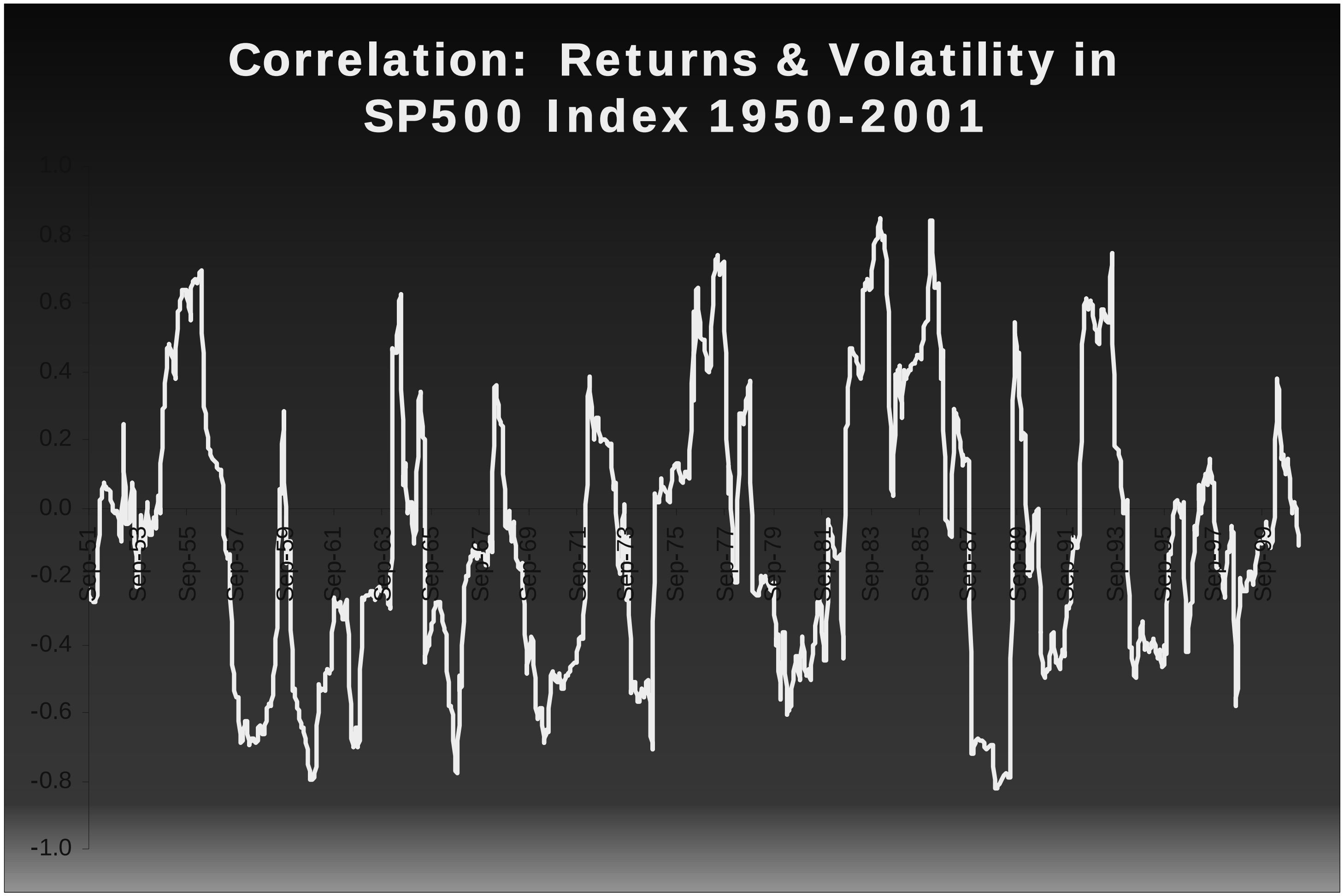
The Volatility Cone



Leverage Effect

- Black (1976)
  - ✓ Stock price changes negatively correlated with volatility
  - ✓ Fixed costs provide a partial explanation
    - Firm with equity and debt becomes more leveraged as stock falls
    - Raises equity returns volatility
  - ✓ Correlation too large to be explained by leverage alone
    - Christe (1982), Schwert (1989)

Correlation in Returns & Volatility



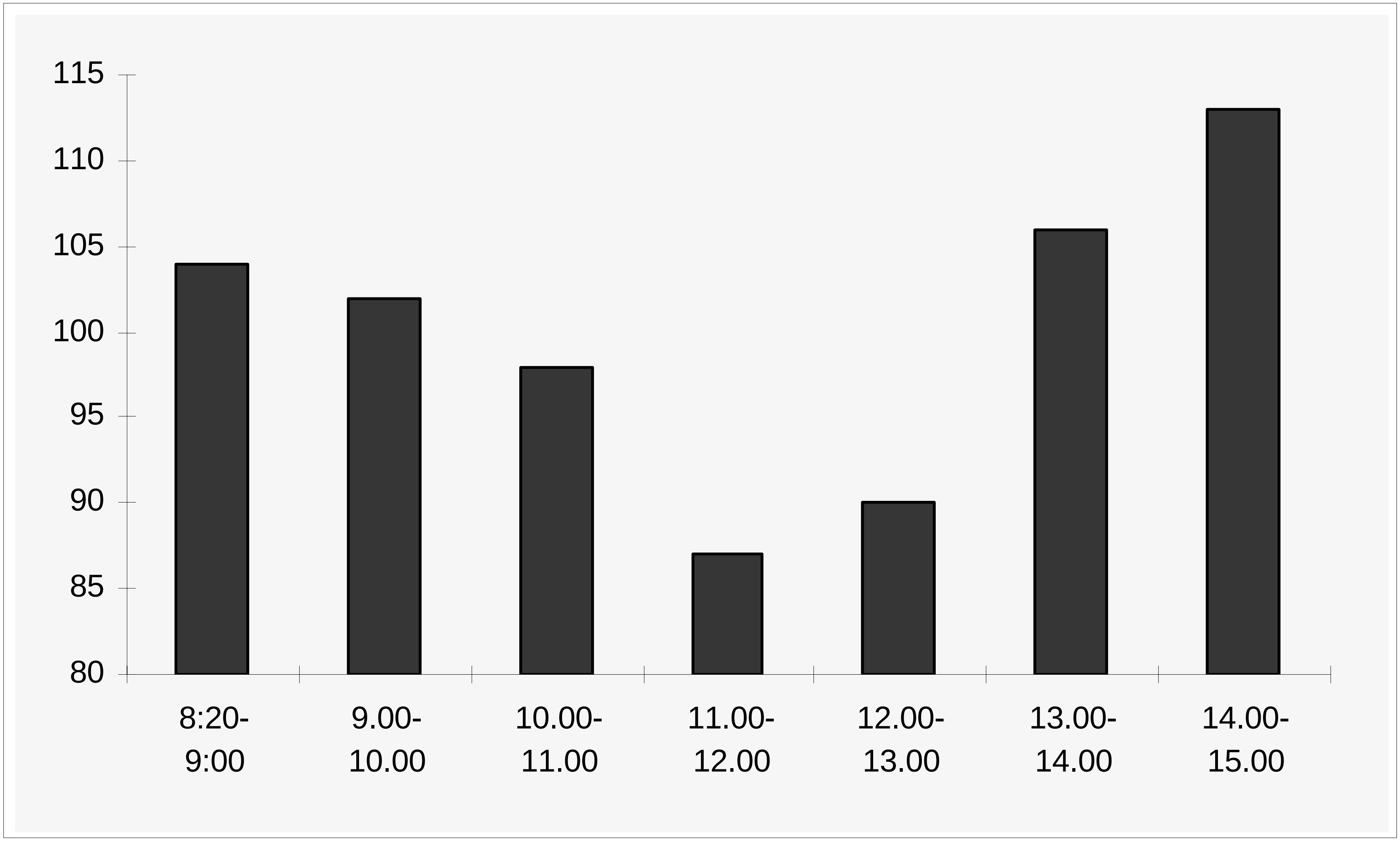
Non Trading Periods

- Trading Causes Volatility
  - ✓ Volatility after weekends is higher
  - ✓ Not 3 x higher (closer to 1.2 x)
  - ✓ French & Roll (1986), NYSE & AMEX
  - ✓ Baillie & Bollerslev (1989) FX

Forecastable Events

- News Announcements
  - ✓Associated with high ex-ante volatility
  - ✓Stock volatility is high around earnings dates
    - Cornell (1978), Patell & Wolfson (1979, 1981)
  - ✓Bond volatility rises on macroeconomic news
    - Harvey & Huang (1991, 1992)
- Seasonal Effects
  - ✓Volatility higher at the open and close
    - Harris (1986), Gerity & Mulherin (1992)

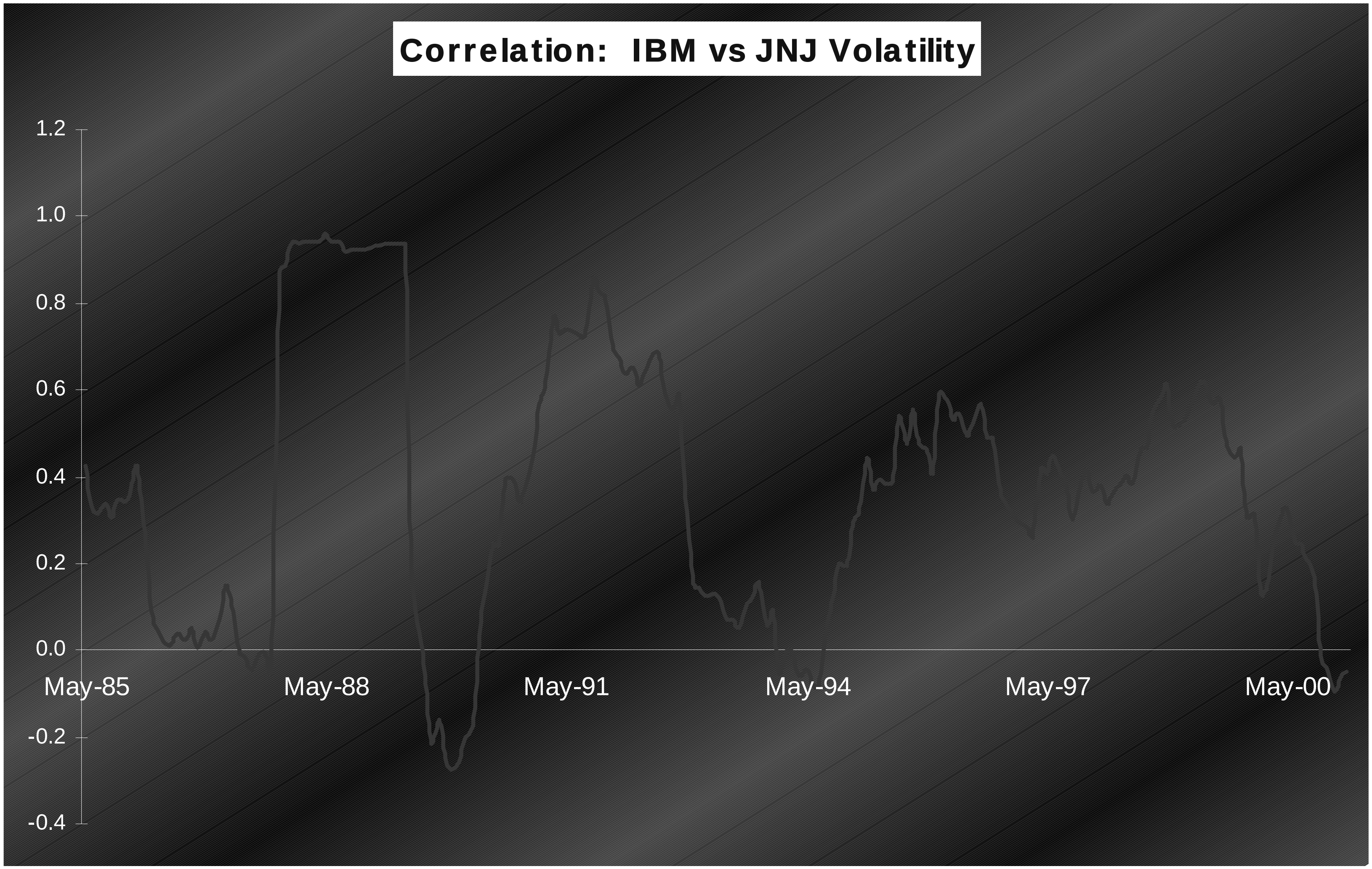
Intra-Day Seasonality in the Bonds



Volatility & Correlation

- Volatility & Serial Correlation in Returns
  - ✓ Strong inverse relationship
    - Stock indices (LeBaron, 1992)
    - FX markets (Kim 1989)
- Co-Movements in Volatility
  - ✓ Volatilities tend to change together
    - Stocks: Black (1976)
    - FX: Diebold & Nerlove (1989)
  - ✓ Also across markets
    - Stock & bond volatilities move together (Schwert, 1989)

Volatility Correlation



Asset Characteristics - Conclusions

- The Bad News
  - ✓iid Gaussian model inappropriate
- The Good News
  - ✓Correlation suggests few common factors may explain variation
  - ✓Stochastic volatility models (GARCH, etc.)

GARCH Models

- Generalized Autoregressive Conditional Heteroscedasticity
  - ✓Time-varying conditional volatility
  - ✓Handles volatility clustering
  - ✓Produces leptokurtic returns distributions
    - Fatter tails than Normal
    - Because changing variance allows for more extreme values
  - ✓Many model variants (>20 !)
  - ✓Very widely applied in finance

Conditional Forecast

- AR(1) process: $y_{t+1} = a_0 + a_1 y_t + \varepsilon_{t+1}$
  - ✓ ε_t is white noise process
    - Constant mean (0), variance σ^2
    - Uncorrelated
- Conditional Forecast
  - ✓ $E_t(y_{t+1}) = a_0 + a_1 y_t$
- Forecast Error Variance
  - ✓ $E_t[y_{t+1} - E_t(y_{t+1})]^2 = E_t[y_{t+1} - (a_0 + a_1 y_t)]^2 = E_t(\varepsilon_{t+1})^2 = \sigma^2$

Unconditional Forecast

- Unconditional Expectation is Constant
  - ✓ $E(y_{t+1}) = a_0 /(1 - a_1)$
    - i.e. the long run mean
- Unconditional Variance is Constant
  - ✓ $E[y_{t+1} - E(y_{t+1})]^2 = E[y_{t+1} - a_0 / (1 - a_1)]^2 = \sigma^2 / (1 - a_1)^2$
  - ✓ Unconditional forecast has greater variance
    - Since $1 / (1 - a_1) > 1$
- Conditional Variance is Constant
  - ✓ $E_t(\varepsilon_{t+1}{}^2) = E_t(y_{t+1} - a_0 + a_1 y_t)^2 = \sigma^2$

Auto-regressive Conditional Heteroskedastic (ARCH) Model

- Suppose conditional variance is *not* constant
- Model conditional variance as an AR(p) process
  - ✓ $\varepsilon_t^2 = \alpha_0 + \alpha_1 (\varepsilon_{t-1})^2 + \alpha_2 (\varepsilon_{t-2})^2 + \ldots + \alpha_q (\varepsilon_{t-q})^2 + v_t$
    - v_t is white noise
- Multiplicative ARCH model (Engle):
  - ✓ $\varepsilon_t^2 = [\alpha_0 + \alpha_1 (\varepsilon_{t-1})^2] v_t^2$
    - is white noise with $\sigma^2_v = 1$
    - ε_t are independent of each other
    - $\alpha_0 > 0$ and $0 < \alpha_1 < 1$

Properties of ARCH

- Unconditional Mean
  - ✓ $E\varepsilon_t = E[v_t(\alpha_0 + \alpha_1 (\varepsilon'_{t-1})^2)^{1/2}]$
    $= Ev_t\, E[(\alpha_0 + \alpha_1 (\varepsilon'_{t-1})^2)^{1/2}] = 0$
- Unconditional Variance
  - ✓ $E\varepsilon_t^2 = E[v_t^2(\alpha_0 + \alpha_1 (\varepsilon'_{t-1})^2)]$
    $= Ev_t^2\; E[\alpha_0 + \alpha_1 (\varepsilon'_{t-1})^2]$
    $= \alpha_0 / (1 - \alpha_1)$
- Covariance
  - ✓ $E[\varepsilon_t \varepsilon_{t-i}] = 0;\; i \neq 0$

Properties of ARCH

- Conditional Mean of Errors
  - ✓ $E(\varepsilon_t / \varepsilon_{t-1}, \varepsilon_{t-2} \ldots) = Ev_t = E(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)^{1/2}] = 0$
- Conditional Variance of Errors
  - ✓ So far, heteroskedasticity has no effect on $\{\varepsilon_t\}$
    - Mean, covariances are zero,variance constant
  - ✓ Impact is on conditional variance
    - $E(\varepsilon_t^2 / \varepsilon_{t-1}, \varepsilon_{t-2} \ldots) = \sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$

Properties of ARCH Series $\{y_t\}$

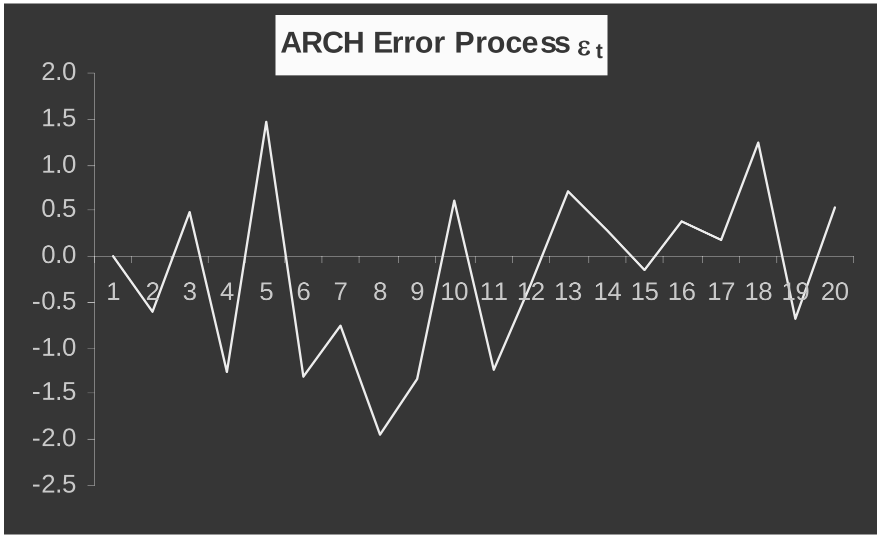
- Conditional Mean: $E_{t-1}(y_t) = a_0 + a_1 y_{t-1}$
- Conditional Variance
  - $Var(y_t / y_{t-1}, y_{t-2}, \ldots) = E_{t-1}(y_t - a_0 - a_1 y_{t-1})^2 = E_{t-1}\varepsilon_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$
  - ✓So *minimum* conditional variance is α_0
    - Since α_1 and $\varepsilon_{t-1}^2 > 0$
- Unconditional Mean
  - ✓$Ey_t = E[a_0 / (1 + a_1) + \Sigma a_1^i \varepsilon_{t-i}] = a_0 / (1 + a_1)$
- Unconditional Variance
  - ✓$Var(y_t) = \Sigma a_1^{2i} var(\varepsilon_{t-I}) = [\alpha_0 / (1 - \alpha_1)] \times [1 / (1 - a_1^2)]$
    - Increasing in both α_0 and $| a_1 |$

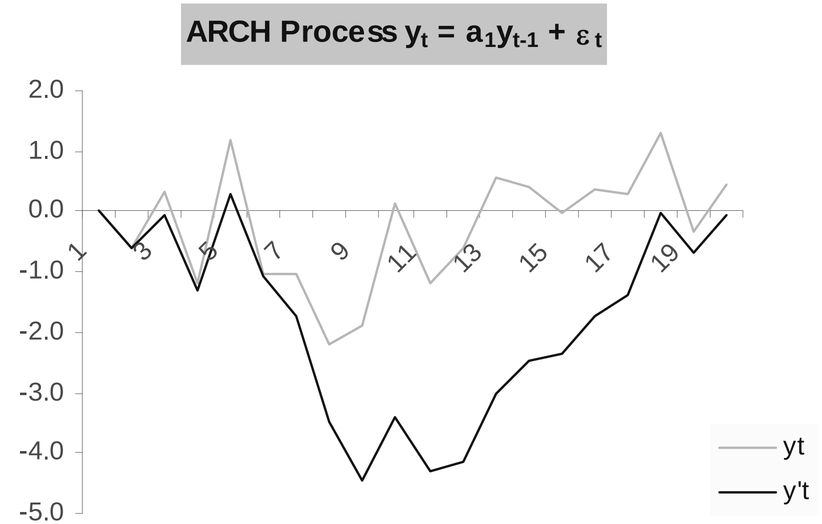
Key Points about ARCH

- Errors Moments
  - ✓Zero mean, covariance, unconditional variance
- Error variance fluctuates
  - ✓For large ε_t , variance of ε_t will be large
  - ✓Periods of tranquility & volatility in {y}
- Errors are not independent
  - ✓Related through second moment
- Parameter values
  - ✓Restricted to ensure variance > 0 and series is stable
    - $\alpha_0 > 0$ and $0 < \alpha_1 < 1$

ARCH Example

- Parameters
  - ✓ $\alpha_0 = 0.3$, $\alpha_1 = 0.9$
  - ✓ $a_1 = 0.25$ & 0.9
- Effects & Interactions
  - ✓ Larger α_1, more persistent are shocks in $\{\varepsilon_t\}$
  - ✓ Larger a_1, more persistent is change in $\{y_t\}$





GARCH Models

- GARCH(p, q)
  - ✓Error Process $\varepsilon_t = v_t \sqrt{h_t}$
  - ✓ $\sigma_v^2 = 1$
  - ✓ $$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}$$
- Error Process $\{\varepsilon_t\}$
  - ✓Conditional mean and variance are zero
  - ✓Conditional variance is h_t

Properties of GARCH

- Disturbances of series $\{y_t\}$ follow ARMA process
  - ✓ARMA(p, q) process in series $\{\varepsilon_t^2\}$

$$E_{t-1}\varepsilon_t^2 = h_t = \alpha_0 + \sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^2 + \sum_{i=1}^{p}\beta_i h_{t-i}$$

- Estimating a GARCH Model
  - ✓Fit ARMA model to series $\{y_t\}$
  - ✓Evaluate sample autocorrelations of squared residuals
    - Should suggest an ARMA(p, q) process in series $\{\varepsilon_t^2\}$

GARCH(1,1) Model

- Model form:

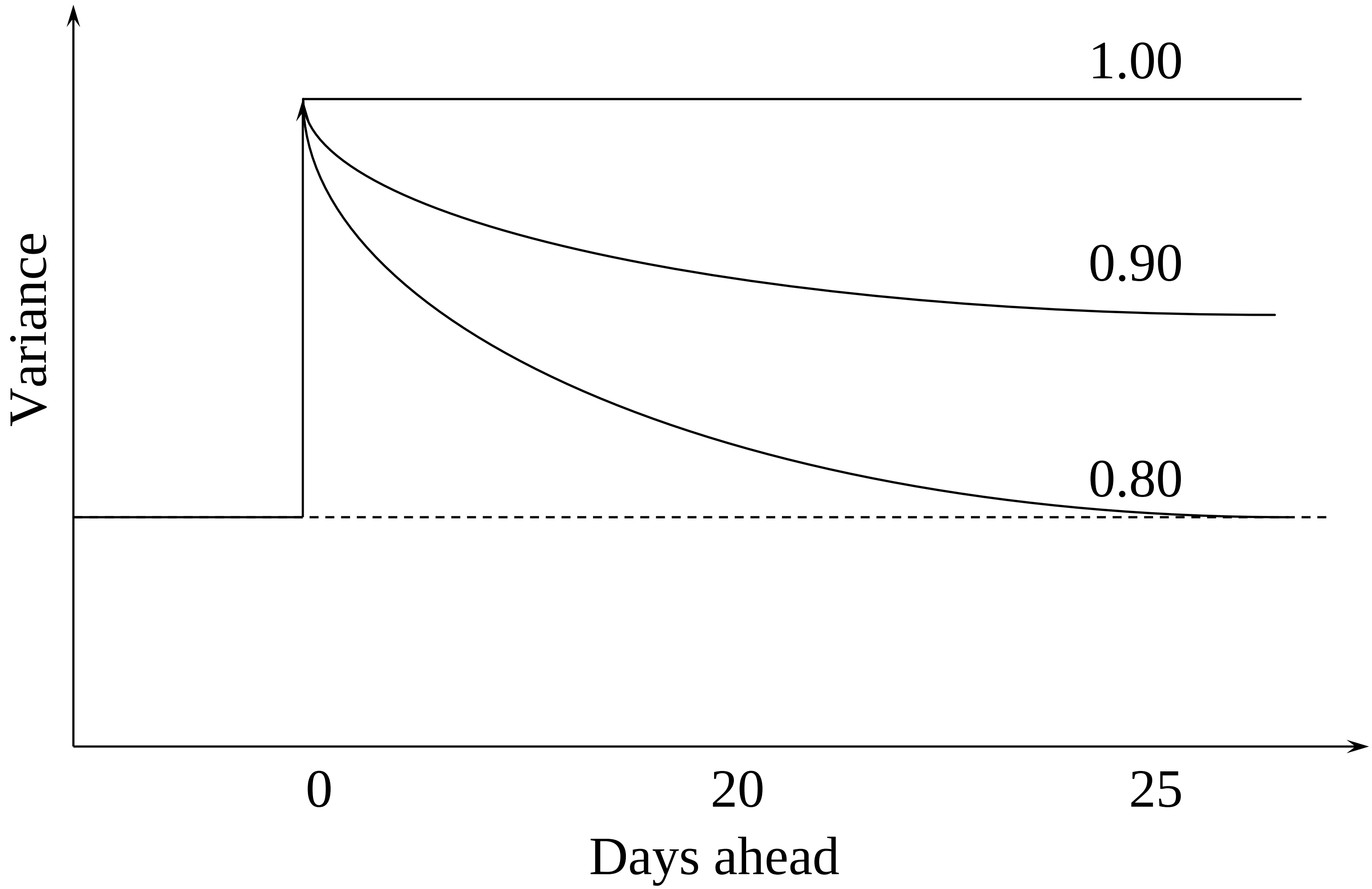
  $$h_t = \alpha_0 + \alpha_1 \varepsilon^2_{t-1} + \beta h_{t-1}$$

- Estimate average, unconditional variance:
  - ✓Set $h_t = h_{t-1} = h$
- Solution: $h = \alpha_0 / (1 - \alpha_1 - \beta)$
  - For stationarity, $\alpha_1 + \beta$ must be < 1
  - Sum $\alpha_1 + \beta$ is known as *persistence*
- Estimation
  - Assume scaled residual $r_t = \varepsilon_t / h_t^{1/2}$ is normally distributed
  - Use numerical optimization to estimate parameters

GARCH Models

Parameter	$/BP	US Stocks	T-Bond
σ (%pa)	11.33	12.02	9.72
α_0	0.00685	0.00233	0.00410
α_1	0.0678	0.0213	0.0132
β	0.9186	0.9740	0.9749
Persistence ($\alpha_1 + \beta$)	0.9864	0.9953	0.9890

Persistence Parameter



GARCH Estimation

- Step 1
  - ✓Fit ARMA model to series $\{y_t\}$
  - ✓Calculate sample variance

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_i^2$$

- Step 2
  - ✓Plot sample autocorrelations of series $\{\varepsilon_t^2\}$

$$\rho(i) = \frac{\sum_{t=i+1}^{T}(\hat{\varepsilon}_t^2 - \hat{\sigma}^2)(\hat{\varepsilon}_{t-i}^2 - \hat{\sigma}^2)}{\sum_{t=i}^{T}(\hat{\varepsilon}_t^2 - \hat{\sigma}^2)^2}$$

GARCH Estimation

- Step 3
  - ✓Test significance of autocorrelations
    - 95% of coefficients should lie in range $\pm 1.96/\sqrt{T}$
    - Ljung-Box test
  - ✓Significant coefficients indicate GARCH errors
- Step 4
  - ✓Fit ARMA(p, q) model to series $\{\varepsilon_t^2\}$
    - Use standard Box-Jenkins methodology

Likelihood Function in Regression

- Simple Linear Regression: $y_t = \beta x_t + \varepsilon_t$
  - ✓ $\varepsilon_t \sim$ IID No$(0, \sigma^2)$
- Likelihood
  - ✓ $L = (-n/2)[\mathrm{Ln}(2\pi) + \mathrm{Ln}(\sigma^2)] - (1/2\sigma^2)\Sigma(y_t - \beta x_t)^2$
- MLE Estimates
  - ✓ $(\sigma')^2 = \Sigma(\varepsilon_t)^2 / n$
  - ✓ $\beta' = \Sigma(x_t y_t) / \Sigma(x_t)^2$
- Standard Error
  - ✓ $\sigma'_\beta = \sigma' / \{\Sigma(x_t - x_{mean})^2\}^{1/2}$

Likelihood Estimation in GARCH

- Now assume $\varepsilon_t^2 = [\alpha_0 + \alpha_1 (\varepsilon_{t-1})^2]\, v_t^2$
  - ✓ Conditional variance $h_t = \alpha_0 + \alpha_1\varepsilon^2_{t-1}$
    - $h_t = \alpha_0 + \alpha_1(y_{t-1} - \beta x_{t-1})^2$
- Likelihood
  - ✓ $L = (-n/2)\text{Ln}(2\pi) - (1/2)\Sigma\, \text{Ln}(h_t) - (1/2)\Sigma\, h_t(y_t - \beta x_t)^2$
    - Maximize L with respect to α_0, α_1 and β
    - Nonlinear model
    - Use search algorithm

Lagrange Multiplier Test

- For ARCH disturbances (Engle 1982)
- Use OLS to fit AR(n) to series $\{y_t\}$
- Form series $\{\varepsilon_t^2\}$
- Estimate regression relationship
  - ✓ $\varepsilon'^2_t = \alpha_0 + \alpha_1 (\varepsilon'_{t-1})^2 + \alpha_2 (\varepsilon'_{t-2})^2 + \ldots + \alpha_q (\varepsilon'_{t-q})^2$
- Testing
  - ✓ If no ARCH effects $\alpha_1 = \alpha_2 = \ldots = \alpha_q = 0$
  - ✓ $TR^2 \sim \chi_q^2$ under the null hypothesis of no ARCH
    - If test statistic is large, reject null of no ARCH

Example: US Inflation

- Bollerslev (1986)
  - ✓ Compared standard ARMA to GARCH
- ARMA $\delta_t = 0.240 + 0.552\delta_{t-1} + 0.177\delta_{t-2} + 0.232\delta_{t-3} - 0.209\delta_{t-4} + \varepsilon_t$
  - ✓ All coefficients significant
  - ✓ No significant ACF or PACF coefficients
  - ✓ However, ACF & PACF of *squared* residuals show significant autocorrelations
- GARCH(1,1)

  $$\delta_t = 0.141 + 0.433\delta_{t-1} + 0.229\delta_{t-2} + 0.349\delta_{t-3} - 0.162\delta_{t-4} + \varepsilon_t$$

  $$h_t = 0.007 + .135\varepsilon_{t-1}^2 + 0.829h_{t-1}$$

  - ✓ Similar forecasts
  - ✓ Changing confidence limits
    - Expand during periods of volatility

ARCH-M Models

- Extension of ARCH
  - ✓ARCH in the mean
  - ✓Engle, Lilien & Robins (1987)
- Applications in Asset Markets
  - ✓Idea: risk premium should be increasing function of conditional variance of returns

ARCH-M Models for Asset Returns

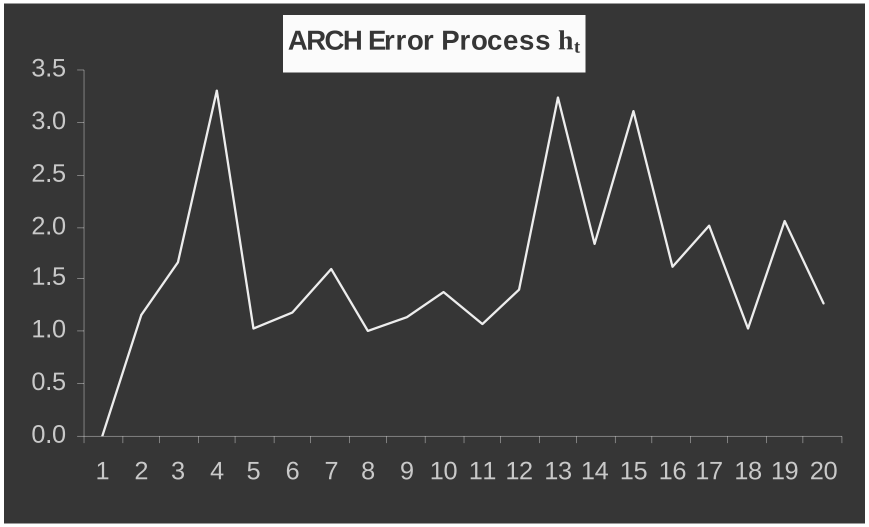
- Assume $y_t = \mu_t + \varepsilon_t$
  - ✓ y_t is the excess return over risk free rate (T-Bills)
  - ✓ μ_t is required risk premium
  - ✓ ε_t is unforecastable shock to excess return
- Assume Risk Premium Follows ARCH Process
  - ✓ $\mu_t = \beta + \delta h_t \, (\delta > 0)$

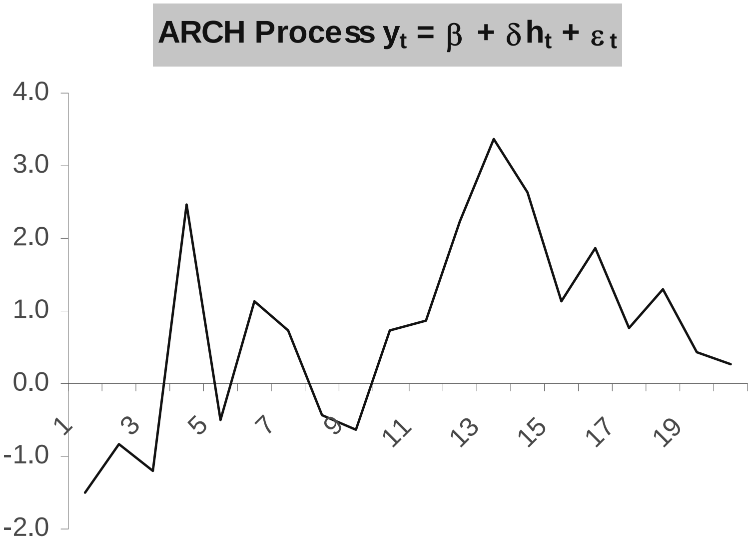
$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$$

ARCH-M Process Example

- Model:

  $y_t = -1 + h_t + \varepsilon_t$





GARCH-M Model

- Process: $y_t = \mu_t + \varepsilon_t$

$$E_{t-1}\varepsilon_t^2 = h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}$$

- Unconditional Variance is Constant

$$Var(\varepsilon_t) = \frac{\alpha_0}{1 - \sum_{i=1}^{q} \alpha_i - \sum_{i=1}^{p} \beta_i} > 0$$

Likelihood Estimation Normal Errors

- Minimize Log(Likelihood) to find optimal parameters

$$Log(L) = -\frac{(n-q)}{2} 2\log(2\pi) - \frac{1}{2}\sum_{t=q+1}^{n} Log(h_t) - \frac{1}{2}\sum_{t=q+1}^{n}\frac{\varepsilon_t^2}{h_t}$$

Likelihood Estimation:Student-t Errors

- Appropriate for Modeling Asset Returns
  - ✓Distribution of standardized residuals $\varepsilon_t / \sqrt{h_t}$ has fatter tales than Normal distribution
    - υ is # degrees of freedom, estimated along with parameters θ

$$Log(L) = \sum_{t=q+1}^{n} L_t(\Theta, \upsilon)$$

$$L_t(\Theta, \upsilon) = -\log\left\{B\left(\frac{\upsilon}{2}, \frac{1}{2}\right)\right\} - \frac{1}{2}Log(\upsilon - 2)$$

$$-\frac{1}{2}Log(h_t) - \left(\frac{\upsilon + 1}{2}\right)Log\left(1 + \frac{\varepsilon_t^2}{h_t(\upsilon - 2)}\right)$$

EGARCH Models

- Exponential GARCH(p, q)

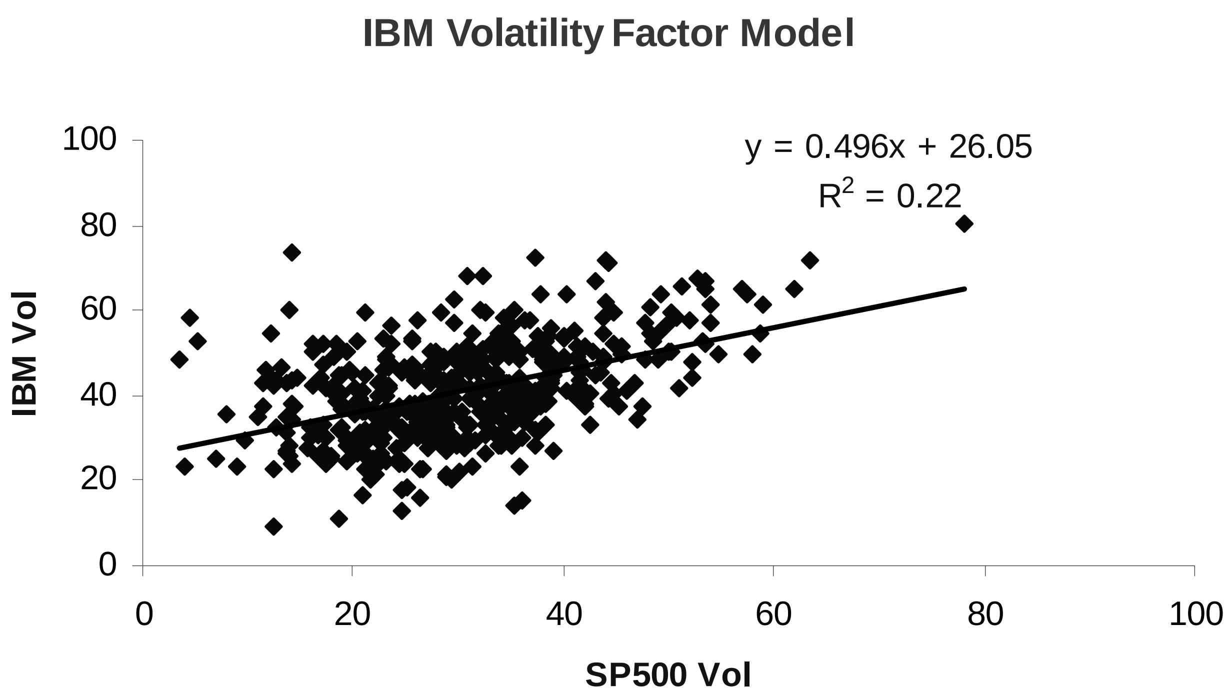
$$Log(h_t) = \alpha_0 + \sum_{i=1}^{q} \alpha_i \left(\frac{\varepsilon_{t-i}}{h_{t-i}^{1/2}} \right) + \sum_{i=1}^{q} \alpha_i^* \left(\left| \frac{\varepsilon_{t-i}}{h_{t-i}^{1/2}} \right| - \varepsilon \right) + \sum_{i=1}^{p} \beta_i h_{t-i}$$

$$\varepsilon = E\left(\left| \frac{\varepsilon_t}{h_t^{1/2}} \right| \right) = \sqrt{\frac{2}{\pi}} \quad if\ \varepsilon_t \approx N(0,1)$$

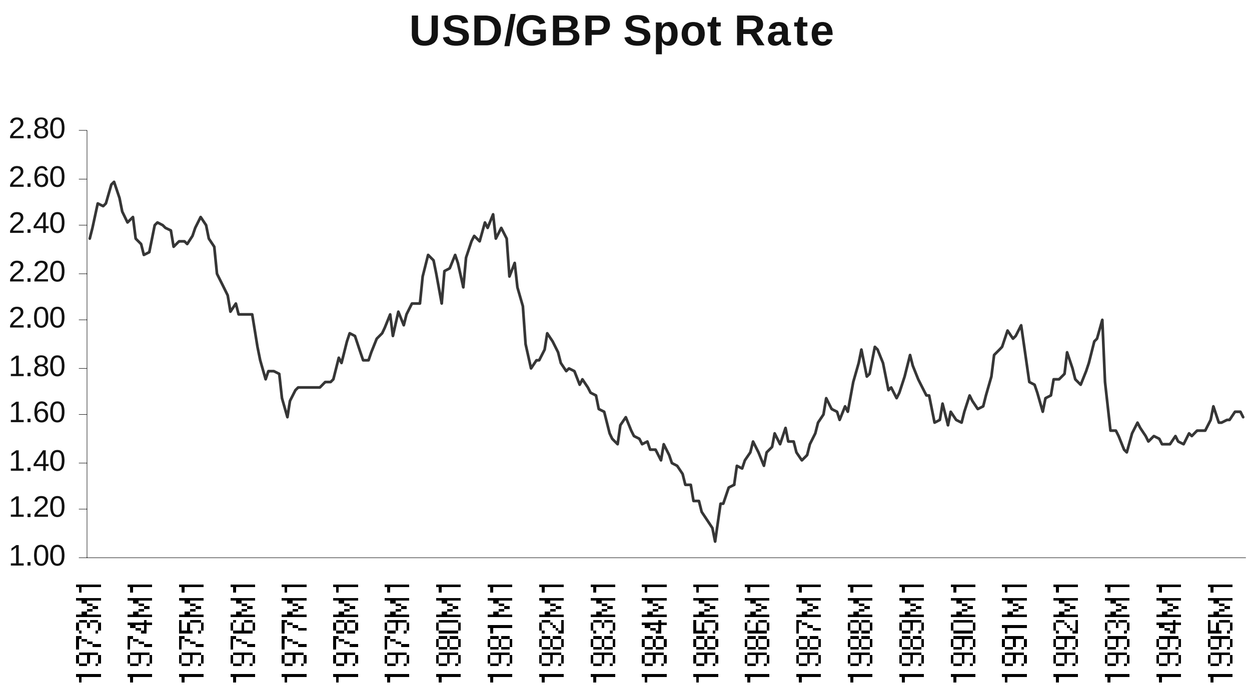
Factor GARCH

- Volatility Equivalent of CAPM

$$\sigma_t^2 = \beta^2 \sigma_{Mt}^2 + \sigma_{\varepsilon t}^2$$



Lab: Estimating an ARCH Model for Cable

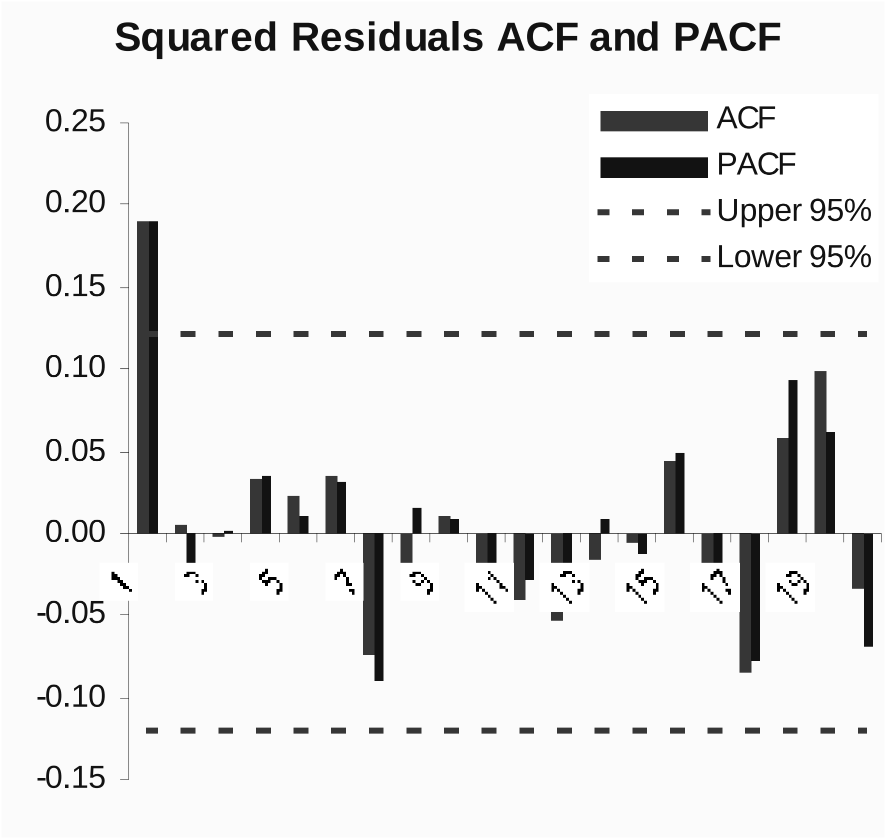


ARCH Model Procedure

- Fit AR(1) Model
  - ✓Form series $\{y_t\} = \Delta Ln(r_t)$
  - ✓$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$
- Examine ACF of Squared Residuals
- Fit ARCH Model to Error Process
  - ✓Estimate model $h_t = \varepsilon_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$
- Lagrange Multiplier Test for ARCH Effects
  - ✓$nR^2 \sim \chi^2(q)$

Solution: ARCH Model for Cable

- Evidence of AR(1) ARCH process:



Solution: ARCH Model for Cable

	MLE
α_0	
α_1	0.3635

Max Likelihood		
AIC	-2616.55	
BIC	-2612.98	
DW	2.07	
R^2	18.0%	p
nR^2	47.452	0.00%

ANOVA	DF	SS	MS	F	p
Model	1	1.04E-05	1.04E-05	57.46	0.00%
Error	261	4.74E-05	1.82E-07		
Total	262	5.79E-05			

GARCH in Research – Equities

- Noh, Engle, Kane (1993/4)
  - ✓ Test of Efficiency of the S&P 500 Index Option Market using Variance Forecasts
    - Straddle trading using GARCH vs IV forecasting
  - ✓ Significant profits generated by GARCH system
    - Return 1.62% /day, after transaction costs
- Highly significant ARCH effects found
  - ✓ Stocks (Engle & Mastufa, 1992)
  - ✓ Indices (Akgiry, 1989)
  - ✓ Futures (Schwert, 1990)

GARCH in Research – Fixed Income

- Weiss (1984)
  - ✓16 different macroeconomic variables
    - Including AAA corporate bind yields
    - Very significant ARCH effects
- Hong (1988) – excess return 3m vs. 1m Tbills
  - ✓GARCH(1,1) model, persistence > 1
  - ✓Engle, Lillien & Robins (1987)
    - Similar results for 6m vs. 3m Tbills

GARCH in Research - Forex

- High Frequency FX Processes
  - ✓ Well described by GARCH(1,1,) Model
    - Hsieh (1989)
    - Taylor (1986)
    - McCurdy & Morgan (1988)
    - Kugler & Lenz (1990)
  - ✓ Less significant for monthly data
    - Diebold (1988), Baillie & Bollerslev (1989)

Summary: GARCH Models

- Major development of last decade
- Now widely used in financial markets forecasting
  - ✓Option volatility
  - ✓Equity & index markets
  - ✓FX markets
- Many financial time series exhibit non-stationarity
  - ✓Due to heteroscedasticity
  - ✓Can be modeled successfully with GARCH
  - ✓Models how the process deals with shocks