Forecasting Financial Markets

Nonlinear Dynamics

Overview

- Fractal distributions
- ARFIMA models
- Chaotic systems
- Phase space
- Correlation integrals
- Lyapunov exponents

Fractal Distributions

- Problems with Gaussian theory of financial markets
  - Non-normal distribution of returns
    - Fat tails
    - Peaked
- Pareto (1897)
  - Found that proportion of people owning huge amounts of wealth was far higher than predicted by (log) normal distribution
  - “Fat-tails”
  - Many examples in nature

Pareto-Levy Distributions

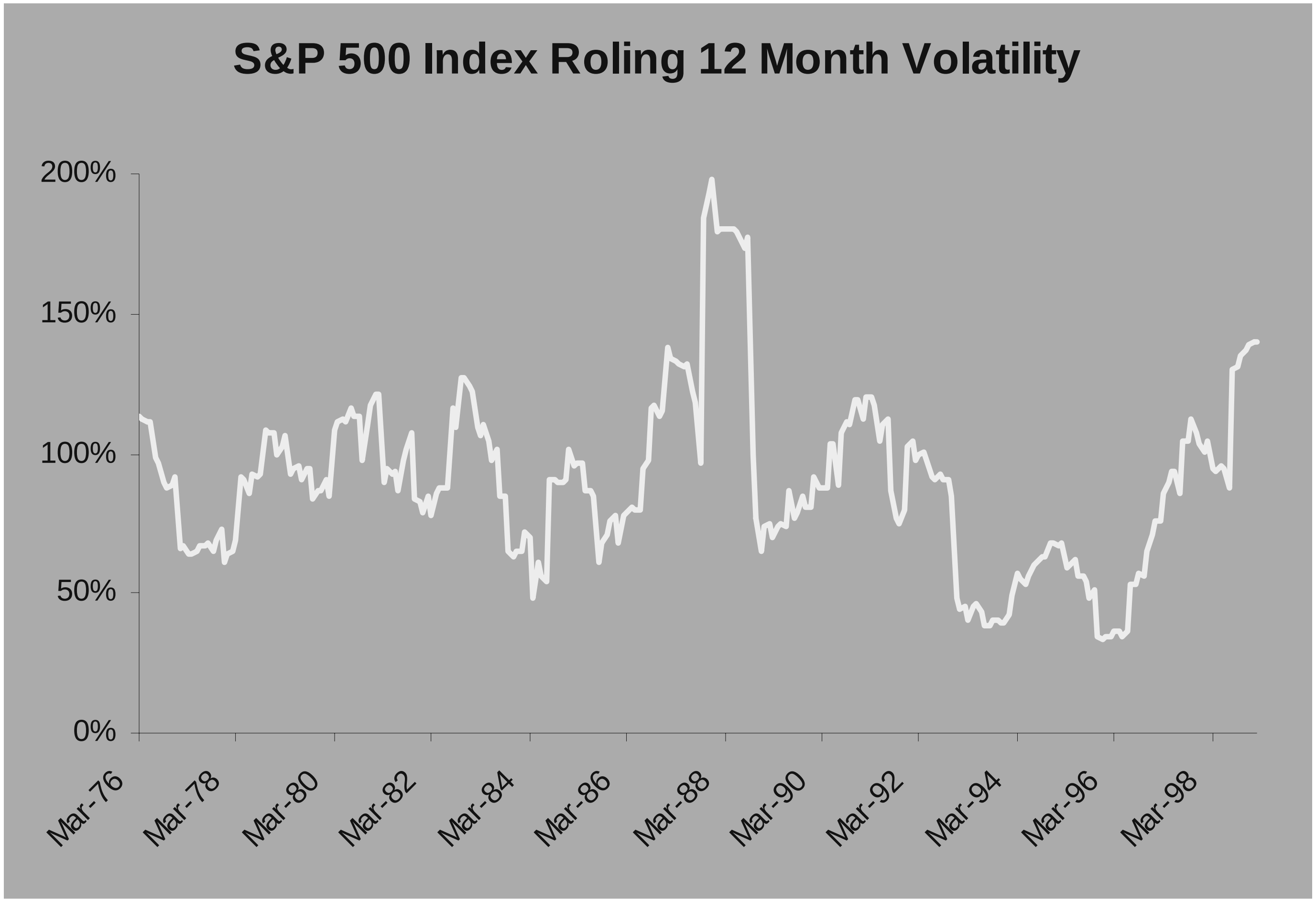
- Levy (1935) generalized Pareto’s law
  - Described family of fat-tailed, high-peak pdf’s
- Pareto-Levy density functions
  - $\mathrm{Ln}[f(t)] = i\delta t - \gamma|t|\alpha(1+i\beta(t/|t|)\tan(\alpha\pi/2)$
  - Parameters
    - α is measure of peakedness
    - β is measure of skewness (range +/- 1)
    - γ is scale parameter
    - δ is location parameter of the mean

Characteristics of Pareto-Levy

- α is fractal dimension of probability space
  - $\alpha = 1 / H$
  - $0 < \alpha < 2$
    - If $\alpha = 2$, ($\beta = 0$, $\gamma = \delta = 1$) distribution is Normal
  - EMH: $\alpha = 2$; FMH: $1 < \alpha < 2$
  - Self-similarity
    - If distribution of daily returns has $\alpha = a$, so will distribution of 5-day returns
  - Variance undefined for $1 <= \alpha < 2$
  - Mean undefined for $\alpha < 1$

Undefined Variance

- Example: Volatility in the S&P 500 Index



ARFIMA Models

- Generalized ARIMA models
  - ARFIMA(p,d,q)
    - Fractional differencing parameter d = H - 0.5
- Models fractal Brownian motion
  - Short memory effects
  - Long memory effects

ARFIMA(0, d, 0)

- No short memory effects
- Long memory depends on parameter d
    - $0 < d < 0.5$: black noise
    - $-0.5 < d < 0$: pink noise
    - D = 0: white noise
    - D = 1: brown noise

ARFIMA(0, d, 0)

- d < 0.5
  - $\{y_t\}$ is stationary
  - Represented as infinite MA process

$$y_t = \sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k}$$

$$\psi_k = \frac{(k+d-1)!}{k!(d-1)!}$$

ARFIMA(0, d, 0)

- d > - 0.5
    - {y_t} is invertible
    - Represented as infinite AR process

$$y_t = \sum_{k=1}^{\infty} \pi_k y_{t-k}$$

$$\pi_k = \frac{(k-d-1)!}{k!(d-1)!}$$

ARMA(0, d, 0)

- Covariance

$$\gamma_k = \frac{(-1)^k (-2d)!}{(k-d)!(-k-d)!}$$

- Correlation

$$\rho_k \sim \frac{(-d)!}{(d-1)!} k^{2d-1} \quad as\ k \to \infty$$

- Partial Autocorrelation

$$\phi_{kk} = \frac{d}{k-d}$$

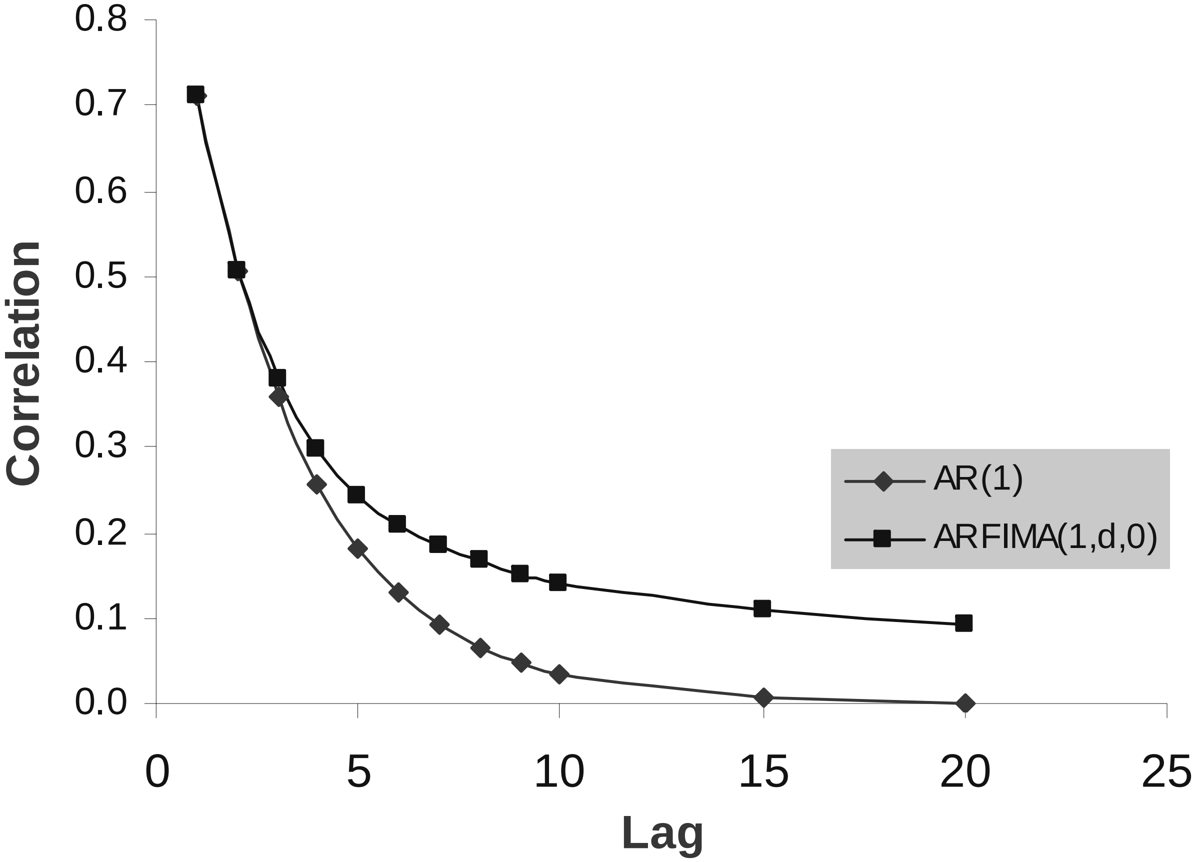
ARFIMA(1, d, 0)

- Process: $(1 - \alpha B)\Delta^d y_t = \varepsilon_t$
  - Combines long and short term memory processes
- Correlation function

$$\rho_k = \frac{(-d)!(1+\alpha)}{(d-1)!(1-\alpha)^2} \times \frac{k^{2d-1}}{F(1,1+d;1-d;\alpha)}$$

- Example: AR(1) vs. ARFIMA(1,d, 0)
  - AR(1): a = 0.711
  - ARFIMA(1, d, 0): d = 0.2, a = 0.5

AR(1) vs. ARFIMA(1, d, 0)



Estimating ARFIMA Models

- Step 1
  - Estimate d in ARIMA(0, d, 0) model $\Delta^d y_t = \varepsilon_t$
    - Use R/S analysis to estimate d
- Step 2
  - Define $u_t = \Delta^d y_t$
  - Use box-Jenkins analysis to fit model $\alpha B u_t = \beta B \varepsilon_t$
- Step 3
  - Define $x_t = (\beta B)^{-1}(\alpha B)\ y_t$
- Step 4: estimate d in model $\Delta^d x_t = \varepsilon_t$
- Repeat steps 2-4 until parameters converge

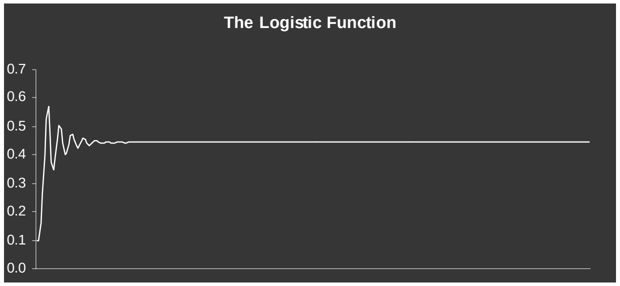
Chaotic Systems

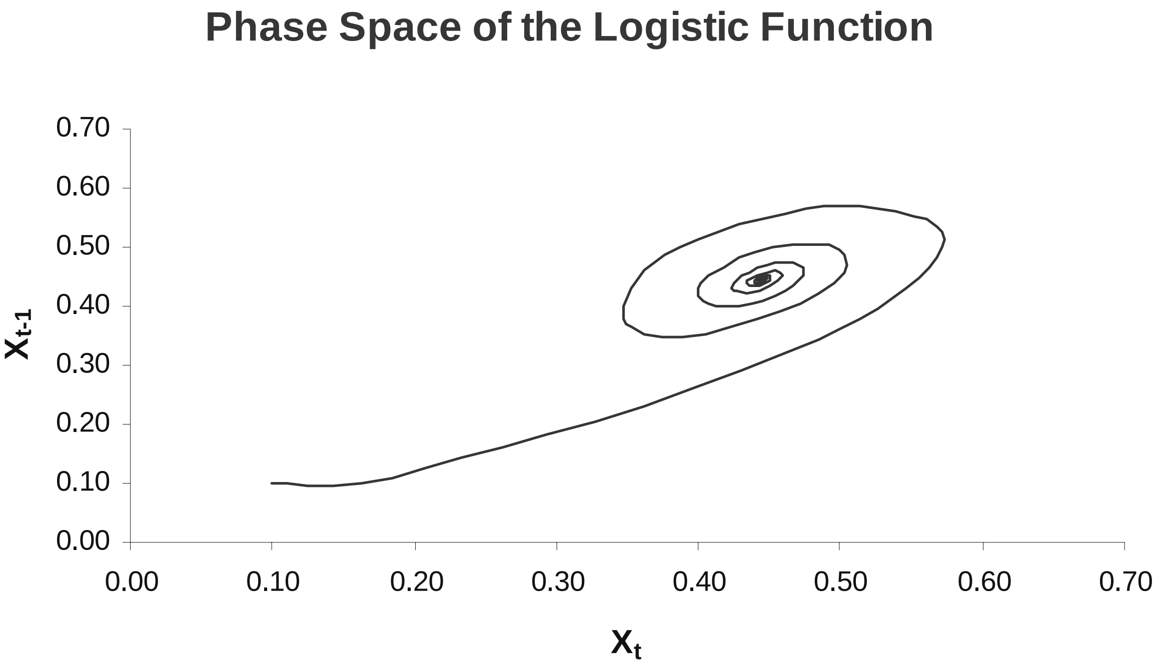
- Charaterized by:
  - Fractal dimension
  - Sensitivity to initial conditions
- Phase space
  - Scatter plot of system variables
  - Allows for visual inspection for patterns
- Attractors
  - Region in phase space where solutions lie
  - Can have Euclidean or fractal dimension

Example: Logistic-Delay Function

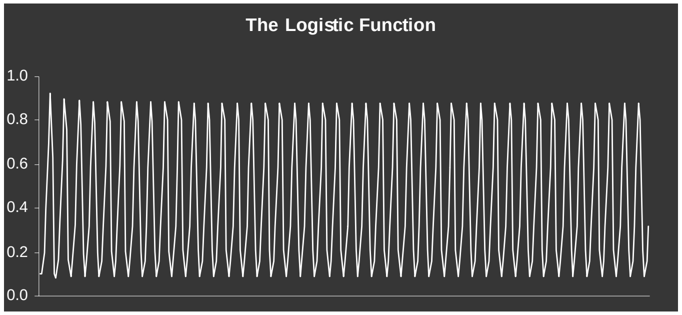
- Equation: $X_t = aX_{t-1} (1-X_{t-2})$
- Attractor dimension
  - Depends on constant a
    - Point attractor (spiral) for smaller values of a
    - Limit cycle (ellipse) as a increases

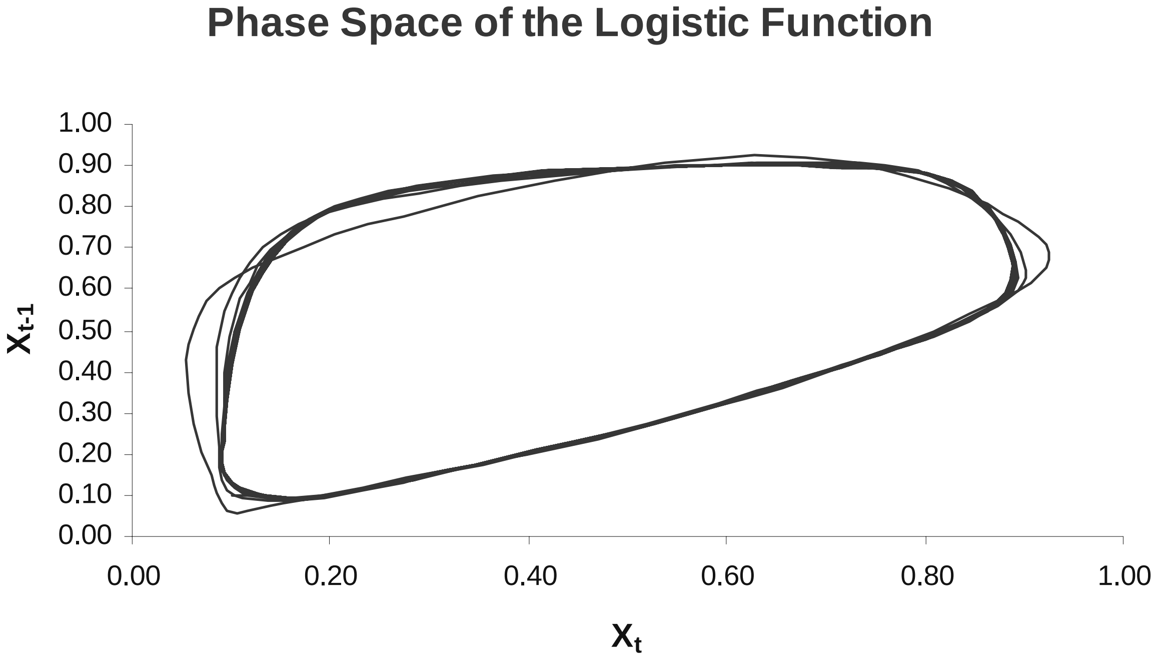
Logistic Function: a = 1.8





Logistic Function: a = 2.2

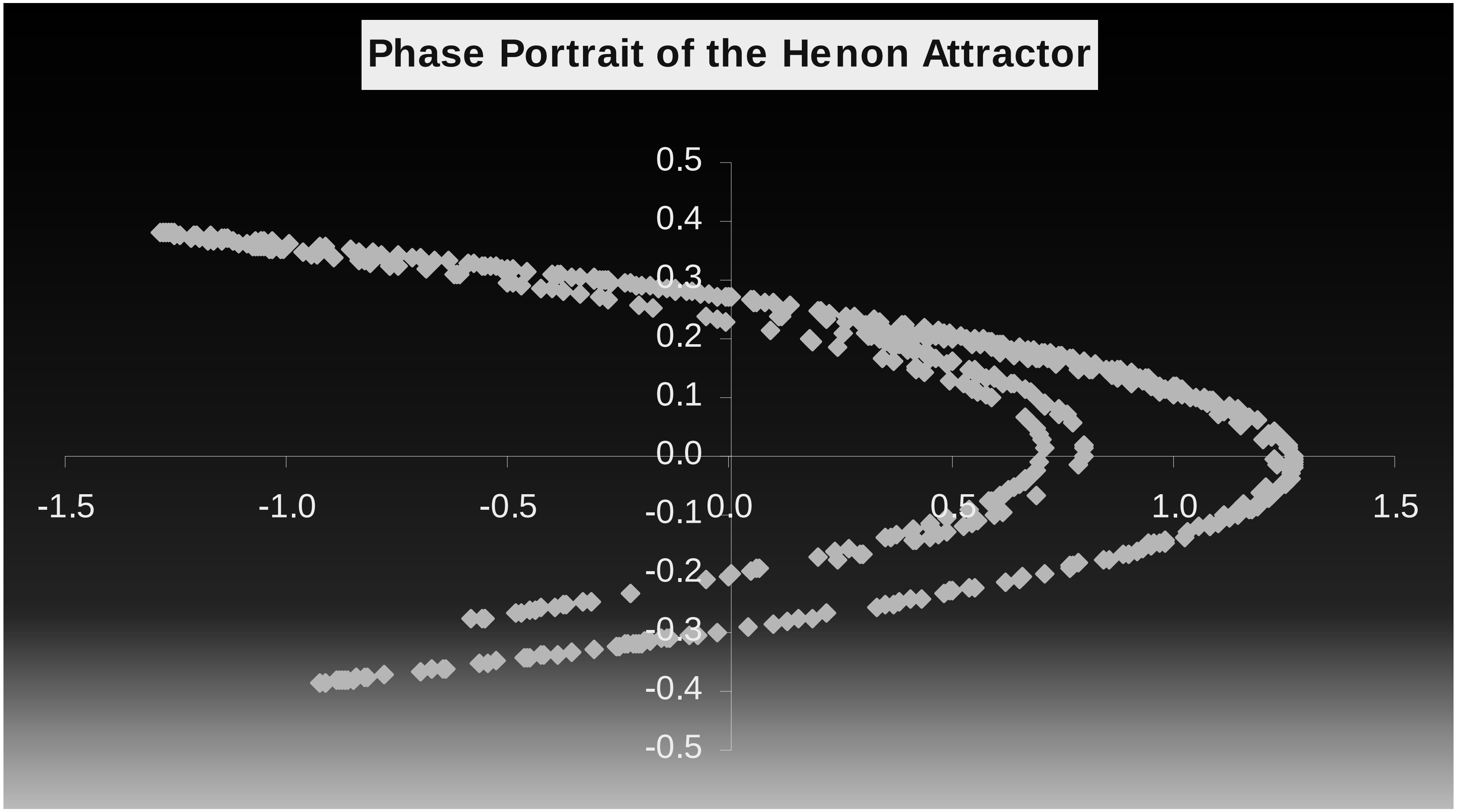




Fractal (“Strange”) Attractors

- Example: Henon attractor
- Equations
  - $x_{t+1} = 1+y_t - ax_t^2$
  - $y_{t+1} = bx_t$
- Phase portrait shows strange attractor
  - Fractal dimension 1.2
  - $1 < D < 2$ indicates presence of 2 variables in system

Henon Attractor



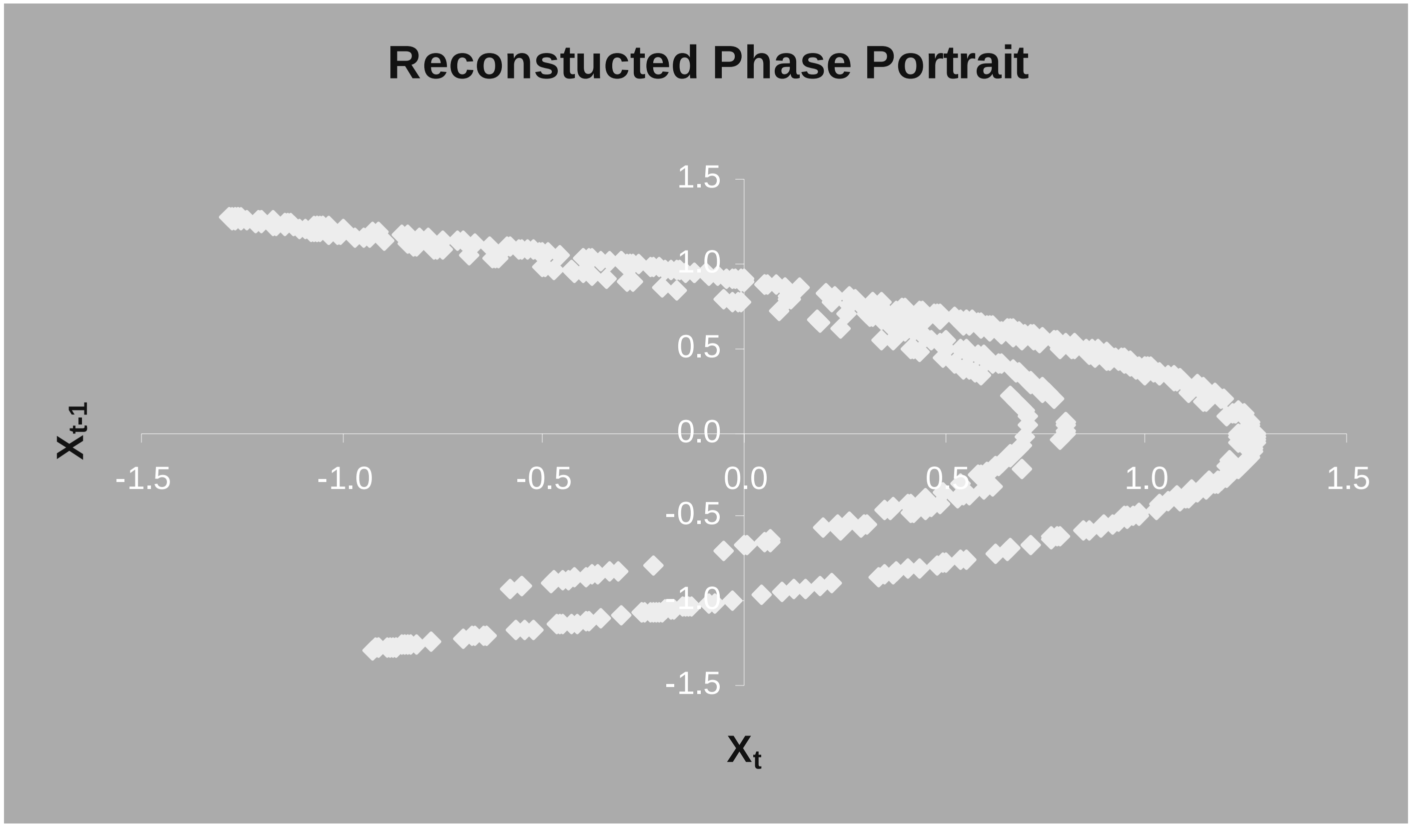
Strange Attractors in the Capital Markets?

- Examine phase portrait of financial time series
  - Fractal attractor would indicate chaotic (i.e. deterministic) process.
  - Dimension of attractor would indicate # of system variables
- Problem:
  - What is dimensionality of phase space?

Constructing Phase Space

- Recall Henon attractor
  - Phase space constructed using scatterplot of two variables X and Y
- Reconstruct phase space
  - Use scatterplot of X_t and X_{t-1}
  - Generates same map
  - Note: constructed using one variable, no equations

Reconstructed Phase Portrait for Henon Attractor



Phase Space Dimensionality

- Ruelle: reconstructed vs. actual phase space
  - Same fractal dimension
  - Same Lyapunov spectrum
- Takens(1981):
  - Can reconstruct phase space by lagging time series for each dimension
- Problem: what time lag to use?
  - i.e. What dimension is attractor?
    - Need to embed it in higher dimension than its own
    - Dimension of attractor does not change when embedded in higher dimension (e.G. A plane in 3-D space still has 2-D)

Determining Embedding Dimensionality

- Wolf: mt = Q
    - M = embedding dimension
    - T = time lag
    - Q = mean orbital period
- Example
    - If period is 48 iterations we require:
        - 2 points lagged 24 iterations in 2-D space
        - 3 points lagged 16 iterations in 3-D space

Fractional Dimensionality of Phase Space

- Time series
  - One variable
  - Dimensionality < 2
- Phase space
  - Includes all variables in system
  - Dimensionality depends on complexity of system
    - May be > 3-D

Correlation Dimension

- Correlation integral
  - Grassberger & Procaccia (1983)
  - Measures probability that pair of points in attractor are within distance R of one another
  - Approximates fractal dimension

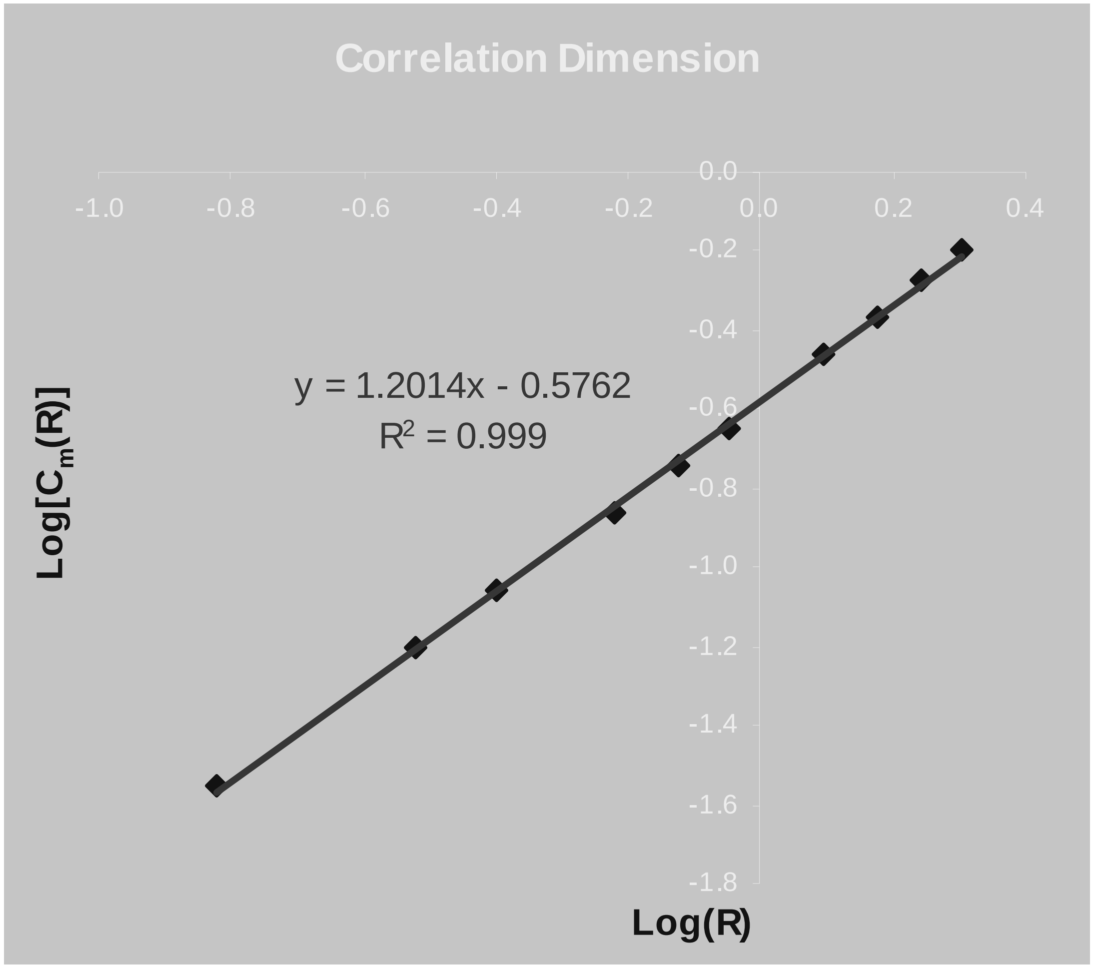
$$C_m(R) = \frac{1}{N^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} Z(R - | X_i - X_j |)$$

  - Z(x) = 1 if x > 0; 0 otherwise

Estimating the Correlation Integral

- As R increases, $C_m(r)$ should increase in proportion to R^D
  - $C_m(R) \sim R^D$
- $\text{Log}[C_m(R)] = \text{const} + D\log(R)$
- Procedure
  - Measure $C_m(r)$ for increasing values of R
  - Log-log plot of $C_m(r)$ vs R
  - OLS estimate of slope is correlation dimension D for embedding dimension m

Correlation Integral of the Henon Attractor



BDS Test for Randomness

- Brock, Dechert, Scheinkman (1987)
  - Lag time series {y_t, t = 1, . . , T} in N lagged series
    - Reconstruct n-dimensional phase space a la Takens
  - $C_N(R,T) \rightarrow C_1(R)^N$ as $T \rightarrow \infty$
  - BDS test statistic

$$W_N(R,T) = |\, C_N(R,T) - C_1(R,T)^N \,| \times \frac{T^{0.5}}{\sigma_N(R,T)}$$

- $\sigma_N(r,t)$ is the SD of the correlation integrals
- W ~ No(0,1)
- For large W, reject the hypothesis that series is random
  - Note will detect both linear and non-linear, so typically use AR(1) residuals to filter out linear effects

BDS Test of Financial Markets

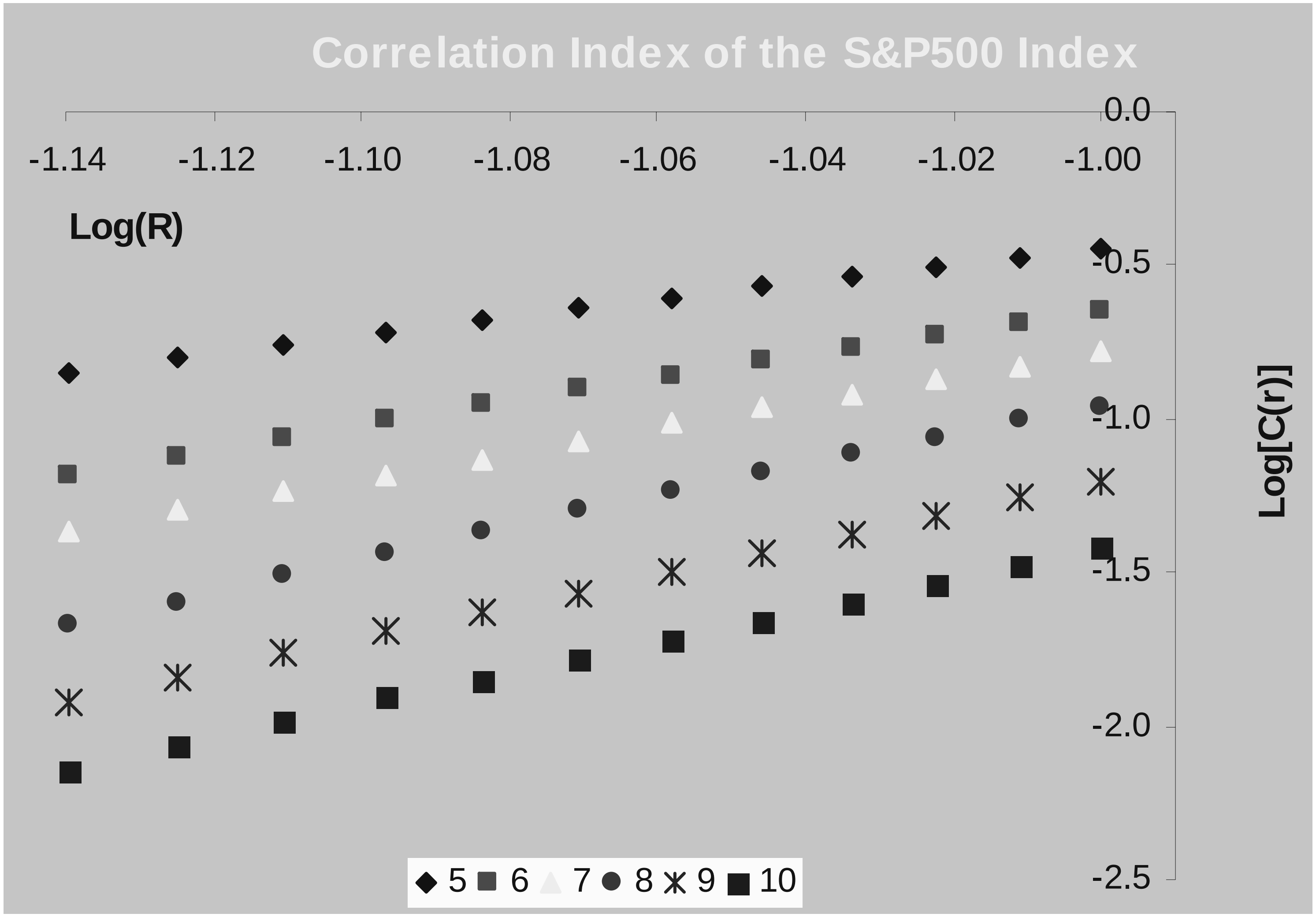
Series	**Dimension**	**W**
Dow (20 day returns)	6	28.72
Yen (daily)	6	116.05
S&P 500 (weekly)	6	23.89

- All tests based on AR(1) residuals of above series
- Sources: Hsieh(1989), LeBaron(1990), Peters (1993)

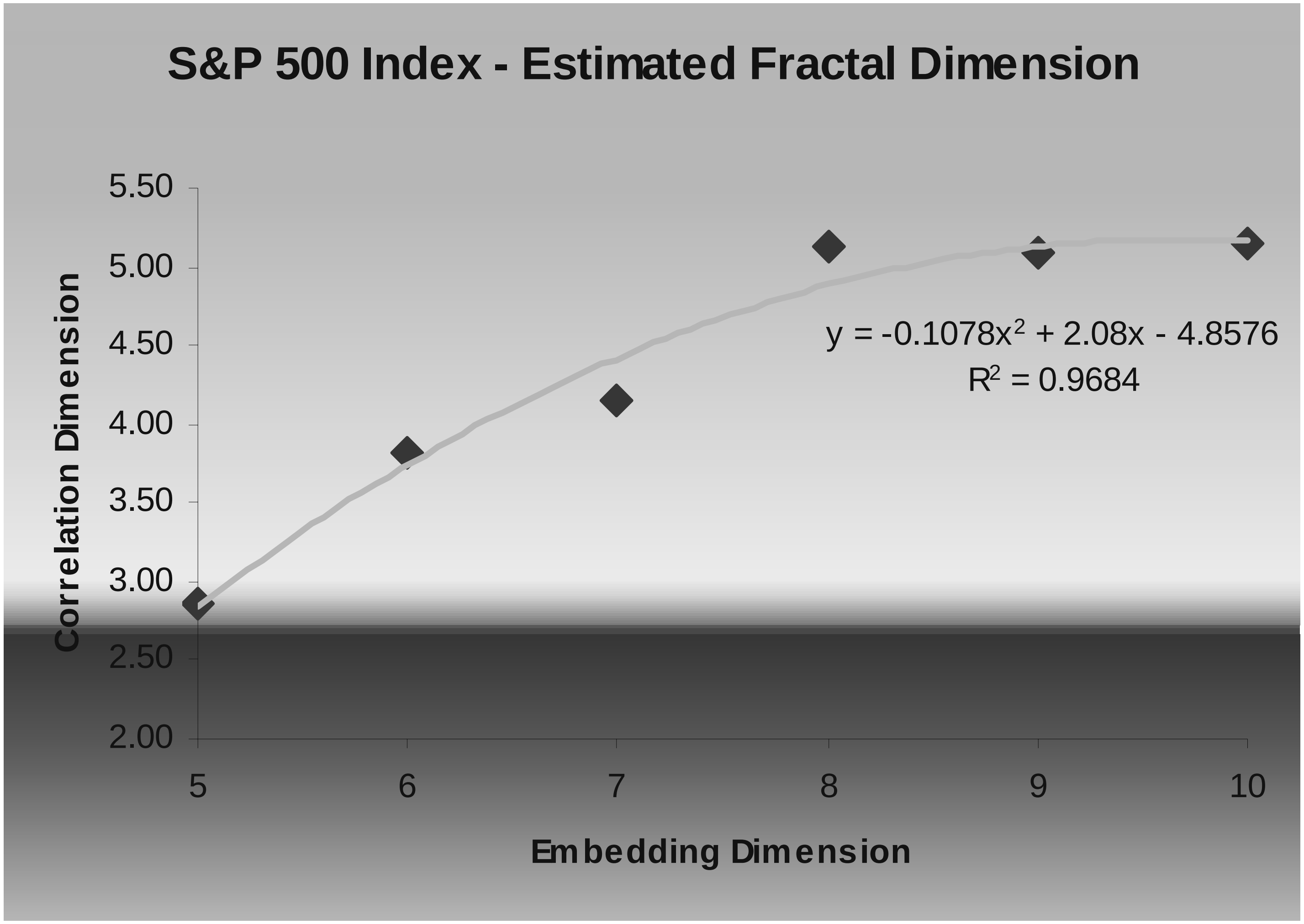
Lab: Estimating the Correlation Dimension for the S&P 500 Index

- Monthly S&P index returns (AR(1) residuals)
  - Cycle estimated at 42 months from R/S analysis
- Estimate correlation dimension
  - Use embedding dimensions m = 5 to 10
  - Lags = Int[42 / m]
- Chart $\log[C_m(r)]$ vs $\log(r)$
    - M = 5 to 10
- Regression analysis
  - Estimate phase space dimensionality D
    - OLS estimate of slope in $\log(r) = D\log[C_m(r)]$

Solution: Estimating the Correlation Dimension for the S&P 500 Index



Solution: Estimating the Correlation Dimension for the S&P 500 Index



Solution: Estimating the Correlation Dimension for the S&P 500 Index

	5	**6**	**7**	**8**	**9**	**10**
D_{EST}	2.86	3.82	4.15	5.12	5.10	5.14
SE	0.029	0.043	0.017	0.058	0.029	0.045
t	98.45	88.13	247.26	88.19	174.43	114.02
p	0.000	0.000	0.000	0.000	0.000	0.000
R^2	99.90%	99.87%	99.98%	99.87%	99.97%	99.92%

- Fractal dimension estimate Stablizes around 5.17
- Concurs with LeBaron & Scheinkman (1986)
  - Daily stock returns had fractal dimension between 5 and 6
- Interpretation
  - 5 or 6 dynamic variables determine S&P index process
  - Extremely complex system, impossible to estimate

Other Studies of Fractal Dimension

- Peters (1991)
  - Criticized LeBaron Study
    - Data insufficiency - would require 10^6 data points to estimate fractal dimension reliably
    - Use of returns not appropriate for study of non-linear effects
  - Used inflation-adjusted prices over 40 year period
- Findings

Equity Index	Est. Dimension
US (S&P500)	2.33
Japan	3.05
Germany	2.41
UK	2.94

Lyapunov Exponents

- Measure of sensitivity to initial conditions
  - How rapidly nearby points in phase space diverge (+ve) or converge (-ve)
  - One exponent for each dimension of phase space
    - Linear dimension grows at rate $2^{L_1 t}$
    - Area grows at rate $2^{(L_1 + L_2)t}$ etc.
- Equation Lyapunov exponent for ith dim. $p_i(t)$

$$L_i = \underset{t \to \infty}{Lim}\left[\frac{1}{t} Log_2\left(\frac{p_i(t)}{p_i(0)}\right)\right]$$

Lyapunov Exponents and Attractors

- Point attractors
  - 3 negative exponents
- Limit cycles
  - 2 negative, one zero exponent
    - 2 dimensions that converge
- 3-D strange attractors
  - One positive, one zero, one negative
    - Positive exponent shows sensitivity to initial conditions
    - Negative exponent causes diverging point to remain in range of attractor

Lyapunov Exponents and the Capital Markets

- Strange attractor?
  - Positive exponent due to technical factors or sentiment
  - Negative exponent due to fundamental value
    - Brings prices back into “reasonable” range

Lyapunov Exponents and Time Series

- Find largest positive Lyapunov exponent L^+
  - Measured in bits per day
  - Means we lose L^+ bits of predictive power / day
- Example: $L^+ = 0.1$
  - We lose 0.1 bits of predictive power / day
  - Suppose we can measure today’s conditions to 1 bit precision
  - Information will lose all value after 1 / 0.1 = 10 days

Estimating the Largest Lyapunov Exponent

- Wolf's algorithm
  - Measures divergence of nearby points in reconstructed phase space
  - Indicates how rate of divergence scales over fixed intervals of time
  - Should converge to L+ if appropriate embedding dimension m and time lag t are chosen

$$L^{+} = \frac{1}{t}\sum_{j=1}^{m} Log_2\left(\frac{L'(t_{j+1})}{L(t_j)}\right)$$

Largest Lyapunov Exponents of International Equity Markets

Equity Market	Lyapunov (bit / month)	Indicated Cycle (months)
S&P500	0.0241	42
UK	0.0280	36
Japan	0.0228	44
Germany	0.0168	60

Source: Peters (1991)

Conclusions

- Long memory process
  - Confirmed by two independent methods of analysis
    - R/S and Lyapunov
- Equity and bond markets - nonlinear systems
  - Aperiodic cycles
    - E.g. Average cycle length 42 months in S&P 500 index
  - Strange attractors
    - Fractal attractor dimension 2.33 (5.17)
  - Fractional noise short term (technical factors?)
  - Chaotic long term (fundamental analysis?)
- Currency markets have no cycle - black noise